§7. Geometrical Consequences

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This lemma implies that R_w is nowhere dense too, so we may apply Theorem 2 to the collection $\{R_w\}$, yielding Theorem 1 (a) for H^3. This completes the proof of Theorem 1.

§ 7. Geometrical Consequences

In this section we summarize some striking geometrical consequences of the existence of large free groups. The following theorem illustrates what can be done with locally commutative actions. Unlike the preceding sections, the results of this section all use the Axiom of Choice. We use $D \Delta E$ to denote $(D \setminus E) \cup (E \setminus D)$.

THEOREM 4. *Suppose a free group,* G, *of rank* $\kappa\,(\kappa \geqslant 2)$ *is locally commutative in its action on* X.

(a) *If (and only if)* $\kappa^\lambda = \kappa = |X|$, *then there is a subset* E *of* X *such that for any* $D \subseteq X$ *with* $|D| \leqslant \lambda$, *there is some* $\sigma \in G$ *such that* $\sigma(E) = E \Delta D$. *In short,* E *is invariant under the addition and deletion of any* λ *points of* X.

(b) X *may be partitioned into* κ *sets,* A_α, $\alpha < \kappa$, *such that each* A_α *is G-equidecomposable with* X *using 2 pieces, i.e. for each* α *there are* $\sigma_\alpha, \tau_\alpha \in G$ *and* $B_\alpha, C_\alpha \subseteq A_\alpha$ *such that* $\{B_\alpha, C_\alpha\}$ *partitions* A_α *and* $\{\sigma_\alpha(B_\alpha), \tau_\alpha(C_\alpha)\}$ *partitions* X. *In short,* X *may be taken apart into pieces which may be rearranged to form* κ *copies of* X.

(c) *There is a subset* E *of* X *such that for any cardinal* λ *satisfying* $3 \leqslant \lambda \leqslant \kappa$, X *may be partitioned into* λ *G-congruent pieces, each of which is G-congruent to* E. *In short,* E *is, simultaneously, a third, a quarter, ..., a* κ*'th part of* X. (If the action is fixed-point free, then $\lambda = 2$ is also permitted — see Theorem 6.)

Parts (b) and (c) of this theorem are applications of a more general fact about locally commutative actions of a free group, which is described following Theorem 6.

Theorem 1 shows that all parts of the preceding theorem, with $\kappa = 2^{\aleph_0}$, apply to S^n, L^n and $H^n\,(n \geqslant 2)$ and $\mathbf{R}^n\,(n \geqslant 3)$, where G is either $G(X)$ or, in the case of L^n, the group of all isometries. Note that, since $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$, part (a) yields a set that is invariant under the addition or deletion of countably many points. Because the existence of large free locally commutative groups was already known in most of these cases, so were the consequences by Theorem 4; only the cases of S^4 and L^4 are new.

Part (a) is due to Mycielski [24]. It is known to be false in $\mathbf{R}^1$, $\mathbf{R}^2$ and S^1 even if one only seeks invariance with respect to the deletion of single points (Sierpiński [37], Straus [38]). Under appropriate (and necessary) assumptions about cardinal arithmetic, part (a) can be used to get sets invariant under the addition and deletion of certain uncountable sets of points. For example, the (consistent) assumption that $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ implies that $(2^{\aleph_0})^{\aleph_1} = 2^{\aleph_0}$, so part (a) is valid with $\kappa = 2^{\aleph_0}$ and $\lambda = \aleph_1$. The proof of Theorem 4 (a) uses the Axiom of Choice, but it is not known whether the set E must necessarily be nonmeasurable.

Part (b) is a refinement of the classical Banach-Tarski Paradox on S^2 along lines first investigated by Robinson [34] and Sierpiński [36]. As stated above, the result is due to Dekker [7], who also proved the following converse.

THEOREM 5. *Suppose* $\kappa \geqslant 2$ *and the action of* G *on* X *satisfies assertion (b) of Theorem 4. Then* G *contains a free subgroup of rank* κ *whose action on* X *is locally commutative; indeed* $\sigma_\alpha^{-1}\tau_\alpha$, $\alpha < \kappa$, *freely generate such a subgroup.*

Work of Banach and von Neumann (see [27]) yields that a solvable group is amenable and whenever an amenable group G acts on X then there exists a finitely additive G-invariant measure μ defined on all subsets of X, with $\mu(X) = 1$. This implies that Theorem 4 (b) is not valid for S^1, $\mathbf{R}^1$ or $\mathbf{R}^2$, even for $\kappa = 2$.

Part (c) of Theorem 4 (Mycielski [22]) is a generalization of an earlier result of Robinson [34], who showed that S^2 may be divided into 3 (or n, if $3 \leqslant n < \aleph_0$) rotationally congruent pieces. It is not clear that Robinson's result requires nonmeasurable pieces, and the following problem (Mycielski [23]) is still unsolved.

Problem. Can S^2 be partitioned into 3 rotationally congruent, Lebesgue measurable sets?

The assertion of 4 (c), however, does necessitate nonmeasurable pieces in S^n and $\mathbf{R}^n$ (for the latter, and for the case of H^n, see § 8). Hence, for the same reasons as for 4 (b), 4 (c) is false in S^1, $\mathbf{R}^1$ and $\mathbf{R}^2$. However, for any $\lambda \leqslant 2^{\aleph_0}$, S^1 may be partitioned into λ pairwise congruent pieces (see [40]). Note that $\lambda = 2$ is omitted from part (c); this is because every element of SO_3, for example, has a fixed point in S^2, therefore S^2 cannot be split into two SO_3-congruent pieces.

Parts (b) and (c) of Theorem 4 are related to the solution of certain systems of congruences. The following theorem (Dekker [7]) shows that a fixed-point free action allows a wide variety of such systems to be solved.

THEOREM 6. *Suppose the action of* G, *a free group of rank* κ, *on* X *is fixed-point free and* $\{\cup\{A_\alpha : \alpha \in L_\beta\} \equiv \cup\{A_\alpha : \alpha \in R_\beta\} : \beta < \kappa\}$ *is a system of* κ *congruences, where each* L_β *and* R_β *is a proper and nonempty subset of* λ. *Then* X *can be partitioned into sets* $A_\alpha, \alpha < \lambda$, *so that each congruence in the system is witnessed by some free generator of* G.

A similar result is true for locally commutative actions, but one has to restrict the systems of congruences to those systems which do not, explicitly or implicity, imply that a set is congruent to its complement. Parts (b) and (c) of Theorem 4 are consequences of this general result. For example, to obtain (b) consider the system.

$$\{A_\alpha \equiv \cup\{A_\beta : \beta < \kappa, \ \beta \neq \alpha + 1\} : \alpha < \kappa, \ \alpha \text{ even}\}$$

and, for $\alpha < \kappa$, α even, let $B_\alpha = A_\alpha$, $C_\alpha = A_{\alpha+1}$.

Because of Theorem 1, Theorem 6, with $\kappa = 2^{\aleph_0}$, applies to S^n and L^n if $n \geqslant 3$ and n is odd, and to H^n and $\mathbf{R}^n$ if $n \geqslant 3$. Moreover, it applies to H^2 if $\kappa = \aleph_0$. Since, as just shown, the conclusion of Theorem 6 implies the assertion of Theorem 4 (b), it follows from Theorem 5 that a partial converse to Theorem 6 is valid: if an action admits a solution to all κ-sized systems of congruences, then G contains a free locally commutative subgroup of rank κ. But the stronger converse to Theorem 6, i.e., the existence of a fixed-point free subgroup, is false. This follows from work of Adams [1], who showed that if the antipodal map from S^n to S^n is available, as it is in SO_{2n} or any O_n, then a locally commutative free group is sufficient to obtain the conclusion of Theorem 6, provided no element of the locally commutative group has -1 as an eigenvalue. This latter condition is clearly satisfied by a free subgroup of SO_3, so Adams' theorem yields the conclusion of Theorem 6 for the action of O_3 on S^2, with $\kappa = 2^{\aleph_0}$. But no free subgroup of O_3 is fixed-point free in its action on S^2.

Because no elements of the locally commutative free subgroups of SO_n constructed by Dekker [7] and Borel [5] have -1 as an eigenvalue, Adams' technique yields the conclusion of Theorem 6, with $\kappa = 2^{\aleph_0}$, for the action of O_{n+1} on S^n, for all $n \geqslant 2$. In fact, any non-Abelian locally commutative free subgroup of SO_3, SO_4 or SO_5 must avoid -1 as an eigenvalue. For SO_3 this is clear since a rotation that sends a point to its antipode must have order 2. Suppose $\sigma, \tau \in SO_5$ freely generate a locally commutative group and some word w has -1 as an eigenvalue. Then this eigenvalue must have multiplicity 2, whence w^2 fixes a 3-dimensional subspace of $\mathbf{R}^5$. Assume w^2 does not begin, on the left, with $\sigma^{\pm 1}$ and let $u = \sigma w^2 \sigma^{-1}$. By freeness, u and w^2 are not powers of a common word; therefore u and w^2 do not commute (see [21, p. 42]). But u also fixes a 3-dimensional subspace, so u and w^2 must share a fixed point on the unit sphere, which contradicts local commutativity. A similar argument works in $\mathbf{R}^4$: choose a basis consisting of two linearly independent fixed points of w^2 and two linearly independent fixed points of u; it follows that u and w^2 commute. These arguments lead to the following question.

Problem. Does SO_6 (or SO_n, $n \geqslant 6$) have a locally commutative free subgroup of rank 2 which contains a transformation having -1 as an eigenvalue?

As an application of Theorem 6, consider the result of Theorem 4 (c). A solution of the following system of $2^{\aleph_0}$ congruences involving A_α, $\alpha < 2^{\aleph_0}$, yields a set E satisfying Theorem 4 (c) for any λ such that $2 \leqslant \lambda \leqslant 2^{\aleph_0}$:

$$A_0 \equiv A_\beta, \qquad \beta < 2^{\aleph_0}$$

$$A_\beta \equiv \cup\{A_\alpha : \beta < \alpha < 2^{\aleph_0}\}, \qquad \beta < 2^{\aleph_0}.$$

Hence, using Adams' result (when necessary), we obtain the following corollary to Theorems 1 and 5.

COROLLARY. *Let X be any of S^n, $n \geqslant 3$, n odd, or $\mathbf{R}^n$ or H^n, with $n \geqslant 3$, and let $G = G(X)$. Or, let X be S^n, $n \geqslant 2$ or L^n, $n \geqslant 3$, n odd, with G being the group of all isometries of X. Then there is a subset E of X such that, for any λ with $2 \leqslant \lambda \leqslant 2^{\aleph_0}$, X may be split into λ sets, each of which is G-congruent to E.*

Because of the anomaly about H^2 discussed in § 6, it is not known whether the conclusion of Theorem 6 is valid in H^2 for some uncountable κ. In particular, we have the following problem, where a set is called a λ'th part of H^2 if H^2 splits into λ sets, each of which is congruent, via $PSL_2(\mathbf{R})$, to the set.

Problem. Does H^2 contain a set which is both a half of H^2 and a $2^{\aleph_0}$'th part of H^2?

Note, however, that because Theorem 6 is valid in H^2 with $\kappa = \aleph_0$ there is a subset of H^2 (indeed, a Borel set; see § 8) that is both a half of H^2 and an $\aleph_0$'th part of H^2; consider the set of congruences preceding the corollary based on the set-variables $\{A_n : n < \aleph_0\}$. Moreover, Theorems 1 (c) and 4 (c) yield a subset that is both a third of H^2 and a $2^{\aleph_0}$'th part of H^2.

§ 8. A Paradoxical Decomposition Using Borel Sets

THEOREM 8. *If* $n \geqslant 2$, *then any system of countably many congruences involving countably many sets (as in Theorem 6) is satisfiable using a partition of* H^n *into Borel sets and isometries.*

Proof. Consider H^2 first, and let F be a free subgroup of $PSL_2(\mathbf{Z})$ whose rank equals the number of congruences to be satisfied; F may be obtained as a subgroup of the group generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and its transpose. Theorem 6 is proved by first constructing, by induction, a partition of F that satisfies the given system using left multiplication in F. Then it is easy to transfer this decomposition to a set on which F's action is fixed-point free by using a choice set for the F-orbits. In general, this requires the Axiom of Choice, and yields nonmeasurable sets. But, because F is a discrete subgroup of $PSL_2(\mathbf{R})$, there is a fundamental region for F's action on H^2. In fact (see [18]) there is a (hyperbolic) polygon such that no two points of the polygon's interior lie in the same F-orbit, and all points in H^2 are in the F-orbit of some point in the closure of the polygon. The boundary of this polygon consists of a countable number of sides (open hyperbolic segments) and vertices. Since F maps vertices to vertices and sides to sides, there is a choice set M for the F-orbits that consists of the interior of the polygon together with some of the vertices and some of the sides. Clearly, M is a Borel set. Now, if B_n is one of the sets of the partition of F, then let $A_n = \cup\{\sigma(M) : \sigma \in B_n\}$. This yields a partition of H^2 into Borel sets A_n which satisfy the given congruences. The result for higher dimensions follows by simple using the standard projection of H^n onto H^2 to define the pieces of a partition of H^n.

COROLLARY. *If* $n \geqslant 2$ *then* H^n *is paradoxical using Borel sets. In fact, there are pairwise disjoint Borel sets,* A_1, A_2, B_1, B_2 *and isometries* $\sigma_1, \sigma_2, \tau_1, \tau_2 \in G(H^n)$ *such that* $H^n = \sigma_1(A_1) \cup \sigma_2(A_2) = \tau_1(B_1) \cup \tau_2(B_2)$. *Moreover, there is a Borel set* E *which is simultaneously a half, a third, ..., an* $\aleph_0$*'th part of* H^2.

This corollary shows that the subsets of H^n provided by parts (b) of (c) of Theorem 4 can be taken to be Borel sets in the case $\kappa = \aleph_0$. This