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$$n = 4^a 9^b, \quad n' = 2 \cdot 4^c 9^d (a, b, c, d \in \mathbf{N}),$$

for which

$$\lambda(n) = 1, \quad \lambda(n') = -1.$$

We therefore have obtained the desired contradiction under the assumption that there exist four consecutive integers  $n \geq 2N_0$ , for which  $\lambda(n) = 1$ . By the part of the theorem already proved, there exist at least three such integers. Therefore (5) holds for some  $m_0 > 2N_0$  with  $n_0 = m_0 + 2$ , and we may now assume that

$$\lambda(m_0 - 1) = \lambda(m_0 + 3) = -1.$$

If  $m_0$  is odd, then this implies

$$\lambda\left(\frac{m_0 - 1}{2}\right) = \lambda\left(\frac{m_0 + 3}{2}\right) = 1, \quad \lambda\left(\frac{m_0 + 1}{2}\right) = -1,$$

so that  $(m_0 + 1)/2 > N_0$  is good, in contradiction to our assumption. But if  $m_0$  is even, then defining  $m_1$  and  $n_1$  by (7), (6) holds for  $i = 1$ , and we have

$$m_1 \geq 2N_0, \quad n_1 - m_1 = \frac{3(m_0 + 2)}{2} - \frac{3m_0}{2} = 3.$$

Thus we are back in the case already treated.

By contradiction, we therefore conclude that (1) has infinitely many solutions for  $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, -1, 1)$ , and the proof of the theorem is complete.

## 5. CONCLUDING REMARKS

In the foregoing proof, the relevant property of the Liouville function was that  $\lambda(n)$  is completely multiplicative and assumes only the values  $\pm 1$ . Besides this, we used only the fact that  $\lambda(2) = \lambda(3) = \lambda(5) = -1$  and (in the proof of the lemma)

$$\lambda(14) = \lambda(16) = 1, \quad \lambda(29) = \lambda(31) = -1.$$

The proof, as it stands, works for any completely multiplicative function  $f(n) = \pm 1$  with these properties. By suitably modifying the proof, it is possible to cover other classes of multiplicative functions as well.

It would be interesting to determine those completely multiplicative functions  $f(n) = \pm 1$ , for which the analogue of the theorem does not hold. Schur [3] proved that if  $f \neq f_{\pm}$ , where

$$f_{\pm}(n) = \begin{cases} (\pm 1)^k & \text{if } n = 3^k m, m \equiv 1 \pmod{3}, \\ -(\pm 1)^k & \text{if } n = 3^k m, m \equiv 2 \pmod{3}, \end{cases}$$

then there exists at least one  $n \geq 1$ , such that

$$f(n) = f(n+1) = f(n+2) = 1.$$

It is likely that under the same hypotheses there are infinitely many such  $n$ . Using arguments similar to those in section 3, one can prove this assertion under the additional hypotheses  $f(2) = 1$  and  $f(3) = -1$ , but the general case seems to be more complicated.

A very plausible conjecture is that the integers  $n$ , for which (1) holds, have positive density. In the case  $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$ , this would follow from an analogous strengthening of the lemma by requiring (2) to hold on a set of positive density. Whereas a very simple argument shows that the equations  $\lambda(n) = \lambda(n+1)$  and  $\lambda(n+1) = \lambda(n-1)$  hold on a set of positive (lower) density (cf. [2]), this argument seems to break down, if  $n$  is required to lie in a prescribed residue class, and so far we have not been able to overcome this difficulty.

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