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$$\partial_x \mapsto \alpha \partial_x - \beta x \partial_t$$

$$x \mapsto \gamma \partial_x \partial_t^{-1} + \delta x$$

$$\partial_t \mapsto \partial_t$$

$$t \mapsto t + \frac{1}{2} \{ \langle \partial_x, {}^t \gamma \alpha \partial_x \rangle \partial_t^{-2} + \langle \partial_x, {}^t \gamma \beta x \rangle \partial_t^{-1} \\ + \langle {}^t \gamma \beta x, \partial_x \rangle \partial_t^{-1} + \langle x, {}^t \delta \beta x \rangle \}.$$

Then we have $\Psi_{g_1} \Psi_{g_2} = \Psi_{g_1 g_2}$.

§ 8. FUNCTORIAL PROPERTIES OF MICRO-DIFFERENTIAL MODULES (See [SKK])

8.1. External Tensor Product.

Let X and Y be complex manifolds and let p_1 and p_2 be the projections $T^*(X \times Y) \rightarrow T^*X$ and $T^*(X \times Y) \rightarrow T^*Y$, respectively. Then $\mathcal{E}_{X \times Y}$ contains $p_1^{-1} \mathcal{E}_X \otimes_{\mathbb{C}} p_2^{-1} \mathcal{E}_Y$ as a subring. For an $\mathcal{E}_X$ -module $\mathcal{M}$ and an $\mathcal{E}_Y$ -module $\mathcal{N}$, we define the $\mathcal{E}_{X \times Y}$ -module $\mathcal{M} \hat{\otimes} \mathcal{N}$ by

$$(8.1.1) \quad \mathcal{M} \hat{\otimes} \mathcal{N} = \mathcal{E}_{X \times Y} \otimes_{p_1^{-1} \mathcal{E}_X \otimes_{\mathbb{C}} p_2^{-1} \mathcal{E}_Y} (p_1^{-1} \mathcal{M} \otimes_{\mathbb{C}} p_2^{-1} \mathcal{N}).$$

Then one can easily see

PROPOSITION 8.1.1.

- (i) $\mathcal{M} \hat{\otimes} \mathcal{N}$ is an exact functor in $\mathcal{M}$ and in $\mathcal{N}$ and $\text{Supp}(\mathcal{M} \hat{\otimes} \mathcal{N}) = \text{Supp } \mathcal{M} \times \text{Supp } \mathcal{N}$.
- (ii) If $\mathcal{M}$ is $\mathcal{E}_X$ -coherent and $\mathcal{N}$ is $\mathcal{E}_Y$ -coherent, then $\mathcal{M} \hat{\otimes} \mathcal{N}$ is $\mathcal{E}_{X \times Y}$ -coherent.

8.2. For a complex submanifold Y of a complex manifold X of codimension l , the sheaf $\varinjlim_m \mathcal{E}xt_{\mathcal{O}_X}^l(\mathcal{O}_X/\mathcal{I}^m, \mathcal{O}_X)$ has a natural structure of $\mathcal{D}_X$ -module,

which is denoted by $\mathcal{B}_{Y|X}$. Here $\mathcal{I}$ is the defining ideal of Y . The homomorphism $\mathcal{O}_Y \rightarrow \mathcal{E}xt_{\mathcal{O}_X}^l(\mathcal{O}_Y, \Omega_X^l) \rightarrow \Omega_X^l \otimes_{\mathcal{O}_X} \mathcal{B}_{Y|X}$ gives the canonical section $c(Y, X)$ of $\Omega_X^l \otimes_{\mathcal{O}_X} \mathcal{B}_{Y|X}$. If we take local coordinates $(x_1, \dots, x_n)$ of X such that Y is defined by $x_1 = \dots = x_l = 0$, then we have

$$\mathcal{B}_{Y|X} \cong \mathcal{D}_X / \sum_{j \leq l} \mathcal{D}_X x_j + \sum_{j > l} \mathcal{D}_X \partial_j.$$

If we denote by δ the canonical generator of the left hand side, then $c(Y, X)$ corresponds to $dx_1 \wedge \dots \wedge dx_l \otimes \delta$. We set

$$\mathcal{C}_{Y|X} = \mathcal{C}_X \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{B}_{Y|X}.$$

Therefore locally we have

$$\mathcal{C}_{Y|X} \cong \mathcal{C}_X / \sum_{j \leq d} \mathcal{C}_X x_j + \sum_{j > d} \mathcal{C}_X \partial_j.$$

Then $\mathcal{C}_{Y|X}$ is a coherent $\mathcal{C}_X$ -module whose support is T_Y^*X .

8.3. For an invertible $\mathcal{O}_X$ -module $\mathcal{L}$, $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1}$ has a natural structure of sheaves of rings, by the composition rule

$$(s \otimes P \otimes s^{\otimes -1}) \circ (s \otimes Q \otimes s^{\otimes -1}) = s \otimes PQ \otimes s^{\otimes -1}$$

for an invertible section s of $\mathcal{L}$ and $P, Q \in \mathcal{C}_X$.

Then the category $\text{Mod}(\mathcal{C}_X)$ of left $\mathcal{C}_X$ -modules and the category $\text{Mod}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1})$ of left $(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1})$ -modules are equivalent by the functor

$$\text{Mod}(\mathcal{C}_X) \ni \mathcal{M} \mapsto \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M} \in \text{Mod}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1}).$$

8.4. Let ω_X be the canonical sheaf on X , i.e. the sheaf of differential forms with top degree. Let a be the antipodal map of T^*X , i.e. the multiplication by -1 . Then we have the anti-ring isomorphism.

$$(8.4.1) \quad \omega_X \otimes_{\mathcal{O}_X} \mathcal{C}_X \otimes_{\mathcal{O}_X} \omega_X^{\otimes -1} \xrightarrow{\sim} a^{-1}\mathcal{C}_X.$$

This homomorphism is given by using a local coordinate system $(x_1, \dots, x_n)$ as follows. For $P = \sum P_j(x, \partial) \in \mathcal{C}_X$ we define $P^* = \sum P_j^*(x, \partial)$, called the formal adjoint of P ([SKK] Chap. II, Th. 1.5.1), by

$$(8.4.2) \quad P_i^*(x, -\xi) = \sum_{\substack{j=l-|\alpha| \\ \alpha \in \mathbb{N}^n}} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_{\xi}^{\alpha} \partial_x^{\alpha} P_j(x, \xi).$$

This is well-defined and satisfies

$$(8.4.3) \quad (P^*)^* = P$$

$$(8.4.4) \quad (PQ)^* = Q^*P^*.$$

Then the isomorphism (8.4.1) is given by

$$(8.4.5) \quad dx \otimes P \otimes (dx)^{\otimes -1} \mapsto P^*$$

where $dx = dx_1 \wedge \dots \wedge dx_n \in \omega_X$. This is independent of coordinate transformations.

8.5. The isomorphism (8.4.1) can be explained as follows. Let Δ_X be the diagonal set of $X \times X$, and let p_j be the j -th projection from $T^*_{\Delta_X}(X \times X)$ to T^*X for $j = 1, 2$. Then the p_j are isomorphisms and $p_2 \circ p_1^{-1} = a$. Let q_j be the j -th projection from $T^*(X \times X)$ to X ($j = 1, 2$). Then $c(\Delta_X, X \times X)$ gives the canonical section of $q_2^{-1}\omega_X \otimes_{q_2^{-1}\mathcal{O}_X} \mathcal{C}_{\Delta_X|X \times X}$. Since $\mathcal{C}_{\Delta_X|X \times X}$ is a

$p_1^{-1}\mathcal{E}_X$ -module, this section gives a homomorphism

$$p_1^{-1}\mathcal{E}_X \rightarrow q_2^{-1}\omega_X \otimes_{q_2^{-1}\mathcal{O}_X} \mathcal{C}_{\Delta_X|X \times X}.$$

It turns out that this is an isomorphism and the right multiplication of $\mathcal{O}_X$ on $\mathcal{E}_X$ corresponds to the $\mathcal{O}_X$ -module structure of $q_2^{-1}\omega_X \otimes_{q_2^{-1}\mathcal{O}_X} \mathcal{C}_{\Delta_X|X \times X}$ via q_2 . Thus we obtain

$$p_1^{-1}(\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}_X \otimes_{\mathcal{O}_X} \omega_X^{\otimes -1}) \xrightarrow{\sim} q_1^{-1}\omega_X \otimes_{q_1^{-1}\mathcal{O}_X} \mathcal{C}_{\Delta_X|X \times X}.$$

This last being isomorphic to $p_2^{-1}\mathcal{E}_X$, we obtain

$$\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}_X \otimes_{\mathcal{O}_X} \omega_X^{\otimes -1} \xrightarrow{\sim} p_1 p_2^{-1} \mathcal{E}_X \simeq a^{-1} \mathcal{E}_X.$$

8.6. By 8.3 and 8.4, if $\mathcal{M}$ is a left $\mathcal{E}_{X|U}$ -module for an open set U of T^*X , then $\omega_X \otimes_{\mathcal{O}_X} a^{-1}\mathcal{M}$ is a right $(\mathcal{E}_{X|aU})$ -module.

8.7. For a left coherent $\mathcal{E}_X$ -module $\mathcal{M}$, $\mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_X)$ is a right coherent $\mathcal{E}_X$ -module. Therefore $\mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_X) \otimes_{\mathcal{O}_X} \omega_X^{\otimes -1}$ is a left $\mathcal{E}_X$ -module by § 8.6.

If $\mathcal{M}$ is holonomic then $\mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_X) = 0$ for $j \neq n = \dim X$ (See [SKK], [K1]). Set $\mathcal{M}^* = \mathcal{E}xt^n_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_X) \otimes_{\mathcal{O}_X} \omega_X^{\otimes -1}$. Then $\mathcal{M}^*$ is also a holonomic $\mathcal{E}_X$ -module.

We call $\mathcal{M}^*$ the *dual system* of $\mathcal{M}$. We have $\mathcal{M}^{**} = \mathcal{M}$, and $\mathcal{M} \mapsto \mathcal{M}^*$ is an exact contravariant functor on the category of holonomic $\mathcal{E}_X$ -modules.

8.8. Let X and Y be complex manifolds, and let $p_1: T^*(X \times Y) \rightarrow T^*X$ and $p_2: T^*(X \times Y) \rightarrow T^*Y$ be the canonical projections. Let p_2^a denote $p_2 \circ a$. Let $\mathcal{K}$ be a left $\mathcal{O}_{X \times Y}$ -module defined on an open subset Ω of $T^*(X \times Y)$. Then, by § 8.6, $\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{K}$ has a structure of $(p_1^{-1}\mathcal{O}_X, p_2^{a-1}\mathcal{O}_Y)$ -bi-module. For an $\mathcal{O}_Y$ -module $\mathcal{N}$,

$$\mathcal{M} = p_{1*}((\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{K}) \otimes_{p_2^{a-1}\mathcal{O}_Y} p_2^{a-1}\mathcal{N})$$

has a structure of $\mathcal{O}_X$ -module. We have the following

THEOREM 8.8.1. *Let Ω , U_X and U_Y be open subsets of $T^*(X \times Y)$, T^*X and T^*Y , respectively. Let $\mathcal{K}$ be a coherent $(\mathcal{O}_{X \times Y}|_{\Omega})$ -module and $\mathcal{N}$ a coherent $(\mathcal{O}_Y|_{U_Y})$ -module. Assume*

(i) $p_1: p_1^{-1}U_X \cap \text{Supp } \mathcal{K} \cap p_2^{a-1}\text{Supp } \mathcal{N} \rightarrow U_X$ is a finite morphism.

Then we have

(a) $\mathcal{F}or_j^{p_2^{a-1}\mathcal{O}_Y}(\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{K}, p_2^{a-1}\mathcal{N})|_{p_1^{-1}U_X} = 0$ for $j \neq 0$.

(b) $\mathcal{M} = p_{1*}((\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{K}) \otimes_{p_2^{a-1}\mathcal{O}_Y} p_2^{a-1}\mathcal{N})|_{U_X}$ is a coherent $\mathcal{O}_X$ -module.

(c) $\text{Supp } \mathcal{M} = U_X \cap p_1(\text{Supp } \mathcal{K} \cap p_2^{a-1}\text{Supp } \mathcal{N})$.

We denote $p_{1*}((\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{K}) \otimes_{p_2^{a-1}\mathcal{O}_Y} p_2^{a-1}\mathcal{N})$ by $\int_Y \mathcal{K} \circ \mathcal{N}$.

8.9. Let $f: X \rightarrow Y$ be a holomorphic map and let Δ_f be the graph of f , i.e. $\{(x, f(x)) \in X \times Y; x \in X\}$, then $\mathcal{K} = \mathcal{O}_{\Delta_f|X \times Y}$ is a coherent $\mathcal{O}_{X \times Y}$ -module whose support is $T_{\Delta_f}^*(X \times Y)$. Now let $\tilde{\omega}$ be the canonical map $X \times_Y T^*Y \rightarrow T^*X$ and ρ the projection $X \times_Y T^*Y$. Then we have the following diagram

$$(8.9.1) \quad \begin{array}{ccccccc} T^*X & \xleftarrow{\tilde{\omega}_f} & X \times_Y T^*Y & \xrightarrow{\rho_f} & T^*Y \\ \text{id} \parallel & & \wr & & \parallel \text{id} \\ T^*X & \xleftarrow{p_1} & T_{\Delta_f}^*(X \times Y) & \xrightarrow{p_2^a} & T^*Y \end{array}$$

We set $\mathcal{E}_{X \rightarrow Y} = \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{C}_{\Delta_f|X \times Y}$ and consider this as a sheaf on $X \times_Y T^*Y$ by the above isomorphism. Then $\mathcal{E}_{X \rightarrow Y}$ is a $(\tilde{\omega}^{-1}\mathcal{E}_X, \rho^{-1}\mathcal{E}_Y)$ -bi-module. For an $\mathcal{E}_Y$ -module $\mathcal{N}$,

$$\int \mathcal{K} \circ \mathcal{N} = \mathbf{R}\tilde{\omega}_* \rho^{-1}(\mathcal{E}_{X \rightarrow Y} \otimes_{\rho^{-1}\mathcal{E}_Y} \rho^{-1}\mathcal{N}).$$

We shall denote this by $f^*\mathcal{N}$ and call it the pull-back of $\mathcal{N}$. Then Theorem 8.8.1 reads as follows.

THEOREM 8.9.1. *Let U_X and U_Y be open subsets of T^*X and T^*Y , respectively. Let $\mathcal{N}$ be a coherent $(\mathcal{E}_Y|_{U_Y})$ -module. Assume*

(i) $\rho_f^{-1}(\text{Supp } \mathcal{N}) \cap \tilde{\omega}_f^{-1}(U_X) \rightarrow U_X$ *is a finite morphism.*

Then we have

(a) $\mathcal{T}or_j^{\rho_f^{-1}\mathcal{E}_Y}(\mathcal{E}_{X \rightarrow Y}, \mathcal{N}) = 0$ *for* $j \neq 0$.

(b) $\mathcal{M} = \tilde{\omega}_{f*}(\mathcal{E}_{X \rightarrow Y} \otimes_{\rho_f^{-1}\mathcal{E}_Y} \rho_f^{-1}\mathcal{N})|_{U_X}$ *is a coherent* $\mathcal{E}_X$ -*module.*

(c) $\text{Supp } \mathcal{M} = \tilde{\omega}_f \rho_f^{-1} \text{Supp } \mathcal{N} \cap U_X$.

8.10. Similarly let $g: Y \rightarrow X$ be a holomorphic map and let Δ_g be the graph of g , i.e. $\{(g(y), y) \in X \times Y; y \in Y\}$. Then we have the isomorphisms

$$(8.10.1) \quad \begin{array}{ccccc} T^*X & \xleftarrow{\rho_g} & Y \times T^*X & \xrightarrow{\tilde{\omega}_g} & T^*Y \\ \parallel & & \wr & & \parallel \text{ id.} \\ T^*X & \xleftarrow{p_1} & T^*_{\Delta_g}(X \times Y) & \xrightarrow[p_2]{\tilde{\omega}_g} & T^*Y. \end{array}$$

We set $\mathcal{E}_{X \leftarrow Y} = \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{C}_{\Delta_g|X \times Y}$ and regard this as a sheaf on $Y \times_X T^*X$. Then $\mathcal{E}_{X \leftarrow Y}$ is a $(\rho^{-1}\mathcal{E}_X, \tilde{\omega}^{-1}\mathcal{E}_Y)$ -bi-module. For an $\mathcal{E}_Y$ -module $\mathcal{N}$ we have

$$\int \mathcal{C}_{\Delta_g|X \times Y} \circ \mathcal{N} = \mathbf{R}\rho_* \tilde{\omega}^{-1}(\mathcal{E}_{X \leftarrow Y} \otimes_{\tilde{\omega}^{-1}\mathcal{E}_Y} \tilde{\omega}^{-1}\mathcal{N}).$$

We shall denote this by $\int_g \mathcal{N}$. Then Theorem 8.8.1 applies to this case and we have

THEOREM 8.10.1. Let U_X and U_Y be open subsets of T^*X and T^*Y , respectively. Let $\mathcal{N}$ be a coherent $(\mathcal{E}_Y|_{U_Y})$ -module. Assume

(i) $\rho_g: \tilde{\omega}_g^{-1}(\text{Supp } \mathcal{N}) \cap \rho_g^{-1}(U_X) \rightarrow U_X$ is a finite morphism.

Then we have

(a) $\mathcal{T}or_j^{\tilde{\omega}_g^{-1}\mathcal{E}_Y}(\mathcal{E}_{X \leftarrow Y}, \tilde{\omega}_g^{-1}\mathcal{N}) = 0$ for $j \neq 0$.

(b) $\mathcal{M} = \rho_{g*}(\mathcal{E}_{X \leftarrow Y} \otimes_{\tilde{\omega}_g^{-1}\mathcal{E}_Y} \tilde{\omega}_g^{-1}\mathcal{N})|_{U_X}$ is a coherent $\mathcal{E}_X|_{U_X}$ -module.

(c) $\text{Supp } \mathcal{M} = \rho_g(\tilde{\omega}_g^{-1} \text{Supp } \mathcal{N} \cap U_X)$.

§ 9. REGULARITY CONDITIONS (See [KK], [K-O])

9.1. Let us recall the notion of regular singularity of ordinary differential equations. Let $P(x, \partial) = \sum_{j \leq m} a_j(x) \partial^j$ be a linear differential operator in one variable x . We assume that the $a_j(x)$ are holomorphic on a neighborhood of $x = 0$. Then we say that the origin 0 is a regular singularity of $Pu = 0$ if

$$(*) \quad \text{ord}_{x=0} a_j(x) \geq \text{ord}_{x=0} a_m(x) - (m-j).$$

Here $\text{ord}_{x=0}$ means the order of the zero. In this case, the local structure of the equation is very simple. In fact, the $\mathcal{D}_X$ -module $\mathcal{D}_X/\mathcal{D}_X P$ is a direct sum of copies of the following modules:

$$\begin{aligned} \mathcal{O}_X &= \mathcal{D}_X/\mathcal{D}_X \partial, \quad \mathcal{B}_{\{0\}|X} = \mathcal{D}_X/\mathcal{D}_X x, \quad \mathcal{D}_X/\mathcal{D}_X (x\partial - \lambda)^{m+1} \quad (\lambda \in \mathbb{C}, m \in \mathbb{N}), \\ &\mathcal{D}_X/\mathcal{D}_X (x\partial)^{m+1} x \quad (m \in \mathbb{N}), \quad \mathcal{D}_X/\mathcal{D}_X \partial (x\partial)^{m+1} \quad (m \in \mathbb{N}). \end{aligned}$$

If we denote by u the canonical generator, then we have $Pu = 0$. By multiplying either a power of ∂ or a power of x , we obtain

$$\sum_{j=0}^N b_j(x) (x\partial)^j u = 0$$

with $b_N(x) = 1$. Hence $\mathcal{F} = \sum_{j=0}^{\infty} \mathcal{O}(x\partial)^j u = \sum_{j=0}^{N-1} \mathcal{O}(x\partial)^j u$ is a coherent $\mathcal{O}$ -submodule of $\mathcal{M}$ which satisfies $(x\partial)\mathcal{F} \subset \mathcal{F}$. We shall generalize this property to the case of several variables.

9.2. Let X be a complex manifold, Ω an open subset of $\overset{\circ}{T}^*X$ and V a closed involutive complex submanifold of Ω . Let us define