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Helped by these statements, Tait gave a list of twenty knots up to ten crossings which are amphicheiral and believed that the list was complete (which it is!).

We conclude this paragraph by recalling a few dates:

a. First proof that knots do exist: H. Tietze in 1908 [Ti] proved that the trefoil is knotted.

b. First proof that non amphicheiral knots do exist: M. Dehn in 1914 [De] proved that the left handed trefoil is not ambient isotopic to the right handed trefoil.

c. First proof that non alternating knots do exist: R. Crowell [Cr] and K. Murasugi [Mu_1] proved in 1957 that the (3, 4) torus knot is non alternating. This result was already stated by C. Bankwitz.

§ 10. L. Kauffman's and K. Murasugi's results

Definition. Let $g(t) \in \mathbf{Z}[t^{\pm 1/2}]$ be a non-zero element:

$$g(t) = \sum_{i=n}^{m} a_i t^i, \quad i \in \frac{1}{2}\mathbf{Z}, \quad a_n \neq 0, \quad a_m \neq 0.$$

Define span $g(t) = m - n$.

In principle span $g(t) \in \frac{1}{2}\mathbf{Z}$. But, if $g(t)$ is the one variable Jones polynomial of an oriented link in S^3, the span of $g(t)$ will actually be an integer. To see that, use induction on complexity, like in § 3.

Definition. Let K be a link in S^3.

K is said to be *splittable* if there exists a 2-sphere $\Sigma \subset S^3$ such that:

1. $\Sigma \cap K = \emptyset$.
2. There is at least one component of K in each connected component of $S^3 - \Sigma$.

THEOREM 10.1. *Let* $K \subset S^3$ *be an oriented unsplittable link. Then:*

$$\text{span } V_K(t) \leqslant c(K).$$

Comments. (i) One can define the number $s(K)$ of split components of K. Then, theorem 10.1 generalizes to:

$$\text{span } V_K(t) \leqslant c(K) + s(K) - 1 .$$

See [Mu_2].

(ii) At first sight, there is something disturbing in this inequality: the polynomial $V_K(t)$ depends on the orientation of K, while the minimal crossing number $c(K)$ does not. But, in fact, span $V_K(t)$ does not depend on orientations, thanks mainly to Jones reversing result.

THEOREM 10.2. *Let L be a connected and oriented link diagram. Suppose L alternating and reduced. Then:*

$$\text{span } V_L(t) = c(L) .$$

Recall that a link is prime if it cannot be decomposed (non trivially) in a connected sum.

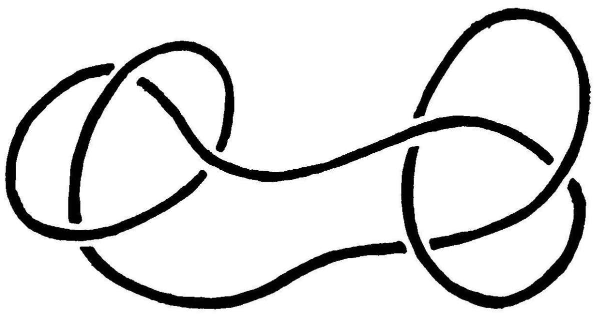
THEOREM 10.3. *Let K be a prime oriented link. Then, for any non alternating diagram L of K one has:*

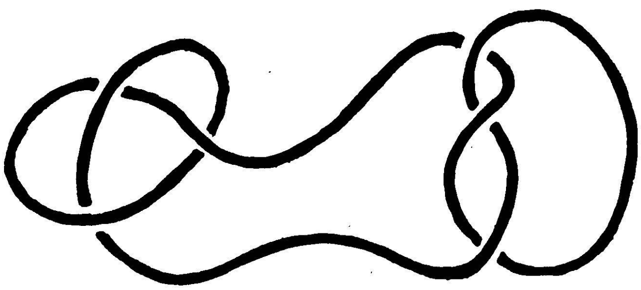
$$\text{span } V_K(t) < c(L) .$$

Comments. (i) We emphazise that the inequality is strict.

(ii) Primeness is necessary, as the following example shows:

Let K be the connected sum of a left-handed and a right-handed trefoil. (This is the so called "square knot".) It is easily proved, for instance by using results of this paper, that $c(K) = 6$. Here are one alternating, and one non-alternating minimal diagrams of K:





As consequences one obtains:

THEOREM 10.4. *Tait conjecture A is true for unsplittable links. (Not only for knots.) The stronger form of conjecture A is true for unsplittable prime links. (For instance for prime knots.)*

This has the following extraordinary consequence concerning knot tabulations, which we illustrate on an example: Suppose you want to prove that the knots 8_{19}, 8_{20} and 8_{21} are non alternating. You may proceed like this:

1. Make the list of knot diagrams with at most 7 crossings (prime or not). Prove the list is exhaustive. (This has already been done by Tait!)
2. Prove that the knots 8_{19}, 8_{20} and 8_{21} are distinct from the preceding ones. Alexander and Jones polynomials may help. Note that the spans of the Jones polynomials for these three knots are strictly smaller than 8.
3. Observe that the knot diagrams 8_{19}, 8_{20} and 8_{21} are non alternating.

Then you know that the knots 8_{19}, 8_{20} and 8_{21} are genuine non-alternating knots!

Proceeding like this step by step (7 crossings, then 8 crossings, etc.), and using computers, M. B. Thistlethwaite can go up to 13 crossings. See [Thi].

By inspection among the 12 695 prime knots with at most 13 crossings, he proves that 6 236 of them are non-alternating. This is a striking example (among others) of the effectiveness of Jones polynomial for proving concrete facts.

THEOREM 10.5. *Conjecture D is true.*

Proof. We know that, for a knot,

$$V_K(t) \in \mathbf{Z}[t^{\pm 1}]. \qquad \text{(i.e. no "halves")}.$$

Moreover $V_K(t) = V_{K^\times}(t^{-1})$.

So, if K is amphicheiral, the span of V_K must be even.

But, for an alternating knot, the span is equal to the minimal crossing number. Q.E.D.

Note. The two references for L. Kauffman and K. Murasugi's results are [Ka_3] and [Mu_2].