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Simon, and Hardt show that there is no singularity for dimension less than six. Recently, L. Simon gave a very good understanding of isolated singularity by studying a neighborhood of the singularity [Si]. He proved that a neighborhood is always described by the graph of a Hölder continuous function defined on the tangent cone.

(iii) *Global problems.*

A lot of problems related to physics, topology and algebraic geometry are global in nature. Hence most of the works of global geometry are related to these subjects. Roughly speaking, "global" means that we study analytic structure over compact spaces. For non-compact space, we request that the structure be complete in some way. For geometric objects defined by these structures, we would like to know their evolution for all time and their asymptotic behavior.

In many ways, the basic questions are

(1) Given a complete analytic structure, how does one deduce global information from local data?

(2) Given the topology of a space, can we put certain analytic structures over this space?

Hence the second question corresponds to an existence theorem in analysis. The first question corresponds to uniqueness. It should be clear from the statements of these questions that understanding of global topology is essential in the treatment of these problems. It turns out that one can turn the argument around and give new theorems in topology. The most recent example is the spectacular achievement of S. Donaldson ([D3, 4]) of applying gauge field theory to understand four dimensional topology. Here the existence theorems was due to Taubes based on the works of K. Uhlenbeck ([T1, 2], [U1, 2]). Normally existence theorems based on pure information of topology is the first step. Once the analytic structure is established, we can deduce topological consequences from the analytic structure. We hope to be able to give some feelings in the following lectures.

§ 1. Eigenvalues and Harmonic Functions

The most fundamental elliptic operator on a Riemannian manifold M is the Laplace operator Δ. If M is compact then Δ has a discrete spectrum. We denote the spectrum (i.e., the eigenvalues of Δ) by $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_i \leqslant \cdots$. It is a basic fact that $\lambda_i \to \infty$ as $i \to \infty$.

There are two basic directions in the study of eigenvalues and they are closely related to each other. The first is the study of the asymptotic behavior of the sequence $\{\lambda_i\}$. The fundamental results can be found in [BGM].

The well-known Weyl's formula gives the first term in the asymptotic expansion for λ_i. It says that

$$\lambda_i \sim C_n i^{\frac{2}{n}} / (\mathrm{vol}\, M)^{\frac{2}{n}} \quad \text{as} \quad i \to \infty ,$$

where C_n is a universal constant depending only on $n = \dim M$.

It is a much harder problem to determine the second term in the asymptotic expansion for λ_i. Ivrii [Iv] has done some significant work in this direction (substantial work was also done by Melrose). His results can be stated briefly as follows. Let M be a compact manifold with boundary $\partial M \neq \emptyset$. We first consider the Dirichlet problem. For a positive real number λ, let $N(\lambda)$ denote the number of eigenvalues (counting multiplicity) which do not exceed λ^2. Under a certain technical assumption related to the set of closed geodesics on M, the following asymptotic formula holds:

$$N(\lambda) = (2\pi)^{-n} W_n(\mathrm{vol}\, M) \cdot \lambda^n - \frac{1}{4}(2\pi)^{-n+1} W_{n-1}(\mathrm{vol}\, \partial M) \cdot \lambda^{n-1} + o(\lambda^{n-1}) .$$

where W_n and W_{n-1} are constants depending only on n. A similar result holds for the Neumann problem. The method Ivrii used was to study the singularity of the fundamental solution of the wave equation $\partial^2 u/\partial t^2 = \Delta u$.

Another problem related to Weyl's formula is the Polyá conjecture. It says the following. If Ω is a bounded domain in $\mathbf{R}^n$, then

$$\lambda_i \geqslant c_n (\mathrm{vol}\, \Omega)^{-2/n}\, i^{2/n}$$

and

$$\mu_i \leqslant C_n (\mathrm{vol}\, \Omega)^{-2/n}\, i^{2/n} .$$

Here $\{\lambda_i\}$ are the eigenvalues for the Dirichlet problem and $\{\mu_i\}$ are the eigenvalues for the Neumann problem.

In [LY1], Li and Yau proved that in the average, the Polyá conjecture is true. The method depends on Fourier transform of the Laplacian. If one can take care of the boundary term for the Fourier transform of high power of the Laplacian, one will be able to settle the conjecture. Furthermore, for any closed manifold M, they also prove

$$\lambda_i \leqslant C_1 + C_2 (i+1)^{2/n} \cdot (\mathrm{vol}\, M)^{-2/n} ,$$

where C_1 and C_2 are constants depending only on m, the diameter of M and a lower bound on the Ricci curvature of M.

The heat kernel, or the fundamental solution of the operator $\partial/\partial t - \Delta$, was a basic tool in understanding eigenvalues of the Laplacian. It often gives an estimate of the eigenvalues with less dependence on the geometry. However, except for the first term in the Weyl's asymptotic estimate, the heat kernel argument is not capable to give information for the lower order asymptotic term at this moment. In any case, a lot of information was obtained in the past by studying the trace of the heat kernel which is $\sum_i e^{-\lambda_i t}$. In particular, one can recover the volume, the total scalar curvature, etc. from this infinite series when $t \to 0$. However, since we have to know all the eigenvalues in order to calculate the asymptotic value, it is not an effective way to recover the invariants. Can one find an *effective* way to calculate the heat invariants? When M is a convex domain, one can actually recover the volume. When M is not convex, the problem is not stable and difficult. In any case, in general one cannot recover all the information about the geometry of the manifold (see [Mi]). C. Gordon and Wilson [GW] found a non-trivial continuous family of metrics on a compact manifold with the same spectrum. However, all the known examples of the manifolds with the same spectrum have the property that they are locally isometric to each other. Is this a generic phenomena?

The second direction in the study of the spectrum is to estimate the low eigenvalues, especially λ_1 for a general manifold by using the mini-max principle. An upper bound was found by Cheng [Ch1] depending only on the diameter of the manifold and a lower bound for the Ricci curvature. Later, Li and Yau [LY1] obtained a lower bound for λ_1 depending on the same data. A sharp lower estimate was found by Zhong [Z]. Both of these estimates have nice applications in geometry. Cheng's theorem implies that a compact manifold, whose Ricci curvature is not less than $n - 1$ and whose diameter is π, is isometric to S^n. The estimate of Li-Yau was used by E. Ruh to provide a new proof of a strengthened version of Gromov's theorem on almost flat manifolds.

It is an interesting and important problem to estimate the gap between eigenvalues. For example, one knows that the multiplicity of $\lambda_1(S^2)$ is less than or equal to three (see [Ch2]). Thus one would like to estimate $\lambda_4 - \lambda_1$. For a convex domain Ω in $\mathbf{R}^n$, $\lambda_1 < \lambda_2$ for the Dirichlet boundary condition. In [SWYY], I. Singer, B. Wong, S. S.-T. Yau and S.-T. Yau gave a lower bound for $\lambda_2 - \lambda_1$ for convex domains. The basic idea of the proof is as follows. Let f_1 and f_2 denote the first two eigenfunctions. Since Ω is convex, the function $u = f_1/f_2$ is well-defined and smooth up to the boundary of Ω. Let $G = |\nabla u|^2 + \lambda(\mu - u)^2$ where $\lambda = \lambda_2 - \lambda_1$ and $\mu = \sup_\Omega u$ by the maximum principle, it is not hard to see that

$$G \leqslant \sup_{\partial\Omega} G = \sup_{\partial\Omega} \lambda(\mu - u)^2 .$$

This implies

$$|\nabla u|^2 + \lambda(\sup u - u)^2 \leqslant \lambda[(\sup u - \inf u)^2 - (\sup u - u)^2]$$

and hence

$$\sqrt{\lambda} \geqslant \frac{|\nabla u|}{\sqrt{(\sup u - \inf u)^2 - (\sup u - u)^2}} .$$

Integrating this inequality along the line segment joining two points where the minimum and maximum of u are obtained, we obtain

$$\lambda_2 - \lambda_1 = \lambda \geqslant \frac{\pi^2}{4d^2},$$

where d is the diameter of Ω. Can one improve the constant of this inequality so that equality actually holds for the interval?

It should be clear that the understanding of eigenvalues depends crucially on understanding the eigenfunctions. A basic part of the eigenfunction is its zero set. It is called the nodal set. Even for two dimensional manifolds, we do not really understand the nodal set. A very famous old problem was to study the nodal line of the second eigenfunction of a convex domain. It was conjectured that it cannot enclose any compact subset of the domain. Recently C. S. Lin [Ln3] proved it under the assumption that the domain has a symmetry. Another interesting question can be posed as follows. Let l_n be the length of the n-th eigenfunction. Can we obtain an asymptotic estimates of l_n? It looks like that it has order $\sqrt{\lambda_n}$ where λ_n is the n-th eigenvalue. The difficult question is to give an upper estimate of l_n.

In the following we will consider the case where M is a complete noncompact manifold. In this case, there should be two theories. An L^2- and an L^∞-theory. We first consider the L^2-theory. The spectrum is then not discrete in general. However, one may still ask: When does $-\Delta$ have an eigenvalue? That is, there exists $f \in L^2(M)$, $f \neq 0$, such that $\Delta f = -\lambda f (\lambda > 0)$. We hope that the following are true.

(1) M doesn't have a pure point spectrum when M is complete with $K \geqslant 0$ (K is the sectional curvature). Escobar [Es] demonstrated this when M is rotationally symmetric outside a compact set.

(2) M does not have an infinite number of eigenvalues when M is complete, simply connected with $-C \leqslant K \leqslant -1$.

(3) M has an infinite number of eigenvalues when M is a complete locally symmetric space with finite volume.

The validity of these conjectures are unknown even when $n = \dim M = 2$. It is known that conjecture 3 is true assuming M is the quotient of a symmetric domain by an arithmetic subgroup.

It is especially interesting to understand the case $\lambda = 0$. In particular, when does M have nonconstant harmonic functions with desired properties, and if so, how many are there?

In [Y5], Yau proved that there are no non-constant L^p-harmonic functions on any complete, noncompact manifold for $1 < p < +\infty$. In particular, there are no L^2-harmonic functions besides constants. Because of this, we will concentrate our attention on positive or bounded harmonic functions. The basic problem we would like to discuss here is to give geometric conditions on M so that the Liouville theorem is either true or false. A result due to Yau [Y7] says that if the Ricci curvature is nonnegative, then the only positive harmonic functions are the constants. Such manifolds may be called *strongly parabolic*.

Very recently, P. Li and Tam [LT] investigated the situation where M has nonnegative sectional curvature outside a compact set. He classified all of the bounded or positive harmonic functions on M. It would be nice if one could replace the above condition by nonnegative Ricci curvature. By using Yau's argument, Donnelly showed that the space of positive harmonic functions is finite dimensional. Such manifolds may be called *parabolic*. It will be interesting to prove that manifolds which are uniformly equivalent to a complete manifold with non-negative Ricci curvature are parabolic. In the other direction, one would like to prove that many manifolds are hyperbolic, i.e., non-parabolic. Anderson [A] and Sullivan [Su] were able to solve the Dirichlet problem for simply connected complete manifolds with curvature bounded by two negative constants. Later, Anderson and Schoen [AS] did some beautiful work on positive harmonic functions on these manifolds. They prove the existence of a C^α-homeomorphism between the Martin boundary and the sphere at infinity $S(\infty)$. The identification of the Martin boundary with the sphere at infinity allows us to begin a systematic study of positive harmonic functions. One can prove that every positive harmonic function u on M can be obtained by the following formula,

$$u = \int_{S(\infty)} K(x, Q) d\mu_Q ,$$

where μ is the unique positive Borel measure on $S(\infty)$, and $K(x, Q)$ is the Poisson kernel. This is the Martin representation formula. Anderson and Schoen were able to study the regularity of K.

One can define harmonic measure on the Martin boundary. It is an important question to study the regularity of this measure. Perhaps a lot of classical facts on harmonic measure for bounded domain have analogues here.

It is not known how to carry through the above theory when the curvature is unbounded from below. It is also not known what the Martin boundary looks like when the curvature is only non-positive. For the case of symmetric domain, there is a well-developed theory. It would be nice to be able to understand symmetric domains through this general framework.

Another important question is to prove that a non-compact complete manifold M is hyperbolic if $\lim_{i\to\infty} \lambda_1(\Omega_i) > 0$ where Ω_i is a compact exhaustion of M.

There are many interesting questions concerning harmonic functions on complete manifolds. A function has polynomial growth of degree k if $|f| \leqslant C(1+r)^{k+\varepsilon}$ for any $\varepsilon > 0$, where $r = d(x, p)$ and p is a fixed point on M. A function has linear growth if it satisfies the previous inequality for $k = 1$. One would like to know a bound on the dimension of harmonic functions with linear or polynomial growth on a complete Riemannian manifold with non-negative Ricci curvature. If M is Kähler, the holomorphic functions with polynomial growth form a ring. In this case, one would like to know when this ring is finitely generated, and when the generators may be chosen to have linear growth. This question is very much related to the following conjecture in Kähler geometry. A complete non-compact Kähler manifold with positive bisectional curvature is biholomorphic to $\mathbf{C}^n$.

Siu-Yau [S-Y2] and Mok-Siu-Yau [MSY] made an attempt to settle this questions by using the L^2-theory of Hörmander. However, the assumption was rather strong. The method was to construct holomorphic functions with slow growth. Recently, Li-Yau [L-Y3] used arguments from elliptic theory to study linear growth holomorphic functions. They made the assumption that the volume growth of the manifold is polynomial with degree $2n$.

In the other direction, it is a major problem to construct bounded holomorphic functions on a complete simplete connected Kähler manifold with strongly negative curvature. In fact, one would like to prove that it is biholomorphic to a bounded domain in $\mathbf{C}^n$ or at least that bounded holomorphic functions separate points of the manifold. It looks like the problem is very much related to a possible generalization of the classical Corona problem to higher dimensional bounded domains.

§ 2. Yamabe's Equation and Conformally Flat Manifolds

Yamabe's equation is a nonlinear elliptic scalar equation related to the conformal deformation of a metric on a Riemannian manifold. Given a metric g_0 with scalar curvature R_0, let g be a metric pointwise conformal to g_0. Then $g = u^{4/(n-2)}g_0$, where $u > 0$ is a smooth function. The scalar curvature R of g is given by the equation

$$L_0 u = -\gamma_0 \Delta_0 u + R_0 u = R u^{\alpha}, \tag{1}$$

where Δ_0 is the Laplacian with respect to g_0, $\gamma_0 = \dfrac{4(n-1)}{n-2}$, $\alpha = \dfrac{n+2}{n-2}$ and $n = \dim M$.

In [Ya], Yamabe asserted that there is always a solution $u > 0$ to equation (1) with $R = \text{const}$. That is to say, any metric on a compact Riemannian manifold is conformally equivalent to a metric with constant scalar curvature. However, his proof contained an error. This was discovered by Trudinger. Moreover, Trudinger [Tr] showed that (1) could be solved for $R = \text{const.}$ provided the lowest eigenvalue λ_1 of the linear operator L_0 is nonpositive.

Let Y be the functional on $L_1^2(M)$ defined by

$$Y = \int_M (\gamma\,|\,\nabla_0 u\,|^2 + R_0 u^2) \Big/ \left(\int_M R u^{\alpha+1}\right)^{\frac{2}{\alpha+1}}$$

where ∇_0 is the gradient with respect to the metric g_0. By a simple computation, one finds that (1) is the Euler-Lagrange equation for the functional Y.

Aubin [Au1] gave a sufficient condition for Y to have a minimum in $L_1^2(M)$. It can be described as follows. Fix $R \equiv 1$ and let $\sigma(g_0)$ be the