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analyticity of the harmonic map f. However, it seems to be difficult to decide which cycles can be represented by continuous images of Kähler manifolds.

§ 6. Canonical Metrics Over Complex Manifolds

Given a complex manifold M, one could like to find "canonical" metrics on M so that one can produce invariants for the complex structure. One natural requirement for canonical metrics is that the totality of them can be parametrized by a finite dimension space and that they be invariant under the group of biholomorphisms.

1. The Bergman, Kobayashi-Royden and Caratheodory metrics

The Bergman metric was first introduced as a natural metric defined on bounded domains in $\mathbf{C}^n$. Later, the definition was generalized to complex manifolds whose canonical bundle K admit sufficiently many sections. For a domain D in $\mathbf{C}^n$, let $H^2(D)$ denote the space of square integrable holomorphic functions of D. Choose an orthonormal basis $\{\phi_i\}$ of this space. Then the Bergman kernel is defined as

$$K(z, w) = \sum_i \phi_i(z)\bar{\phi}_i(w).$$

Notice that the definition of the Bergman kernel is independent of the choice of orthonormal basis. Moreover, K is holomorphic in the variables z and $\bar{w}$.

We can now define the Bergman metric by

$$ds^2 = \sum \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K(z, z)\, dz_i \otimes d\bar{z}_j.$$

The naturality of the Bergman metric can easily be seen from the definition of the Bergman kernel. Let D_1 and D_2 be two domains in $\mathbf{C}^n$, and $K_1(z, w)$ and $K_2(z', w')$ their respective Bergman kernels. If $F: D_1 \to D_2$ is a biholomorphism, then K_1 and K_2 are related by the formula

$$K_1(z, w) = K_2(f(z), f(w)) \det\left(\frac{\partial F}{\partial z}\right) \overline{\det\left(\frac{\partial F}{\partial w}\right)}.$$

If the canonical bundle K of M admits enough global, square integrable sections, we can choose an orthonormal basis $\{\phi_i\}$ of sections which will give rise to an embedding $F: M \to \mathbf{CP}^k$. The pull-back metric $F^*(ds^2)$ is the Bergman metric of M. This definition agrees with the previous definition of the Bergman metric when M is a complex domain because any holomorphic function over D can be though of as a section of K.

Intuitively speaking, a complete understanding of the Bergman metric would give us a clear picture of the geometry of the automorphisms of a domain; it would also provide us with a lot of invariants of the domain. In the past few years there has been a lot of progress based on Fefferman's work [Fe]. Fefferman looked at the asymptotic behavior of $K(z, z)$ near the boundary of a domain. Roughly, he proved that the Bergman kernel has the following expansion along the diagonal.

$$K(z, z) = \phi(z)/\Psi^{n+1}(z) + \tilde{\phi}(z) \log \Psi(z)$$

where $\phi, \tilde{\phi} \in C^\infty(D)$, $\phi\,|_{\partial D} = 0$, and Ψ is the defining function for the domain D.

Moreover, near the boundary we have

$$K(z, w) = \phi(z, w)/\Psi^{n+1}(z, w) + \tilde{\phi}(z, w) \log \Psi(z, w)$$

where $\phi(z, w)$, $\tilde{\phi}(z, w)$ and $\Psi(z, w)$ are extensions of ϕ, $\tilde{\phi}$ and Ψ, respectively, which satisfy certain conditions.

One would actually like to know more about the boundary behavior of the Bergman kernel and metric, the behavior of the curvature of the metric, and other related geometric properties of the metric when Ω is not smooth. Let Ω be a manifold and ds^2_Ω the Bergman metric. If Ω admits a properly discontinuous group of automorphisms we can consider the quotient manifold Ω/Γ and pull-back its Bergman metric $ds^2_{\Omega/\Gamma}$ to Ω. Kazhdan [Kz] proved that if the discrete automorphism group Γ of Ω has a filtration $\Gamma \supseteq \Gamma_1 \supseteq \cdots \supseteq \Gamma_n \supseteq \cdots$ with $[\Gamma_i, \Gamma_{i+1}] < \infty$ and $\bigcap_i \Gamma_i = (1)$, then the pull-backs of the Bergman metrics $ds_i^2 = ds^2_{\Omega/\Gamma_i}$ will converge on Ω to the Bergman metric ds^2_Ω of Ω.

Another interesting direction is to look at the global sections of the powers of the canonical bundle. Consider $H^0(M, K^r)$ for r sufficiently large; a choice of basis gives a map $\phi_r: M \to \mathbf{P}(H^2(M, K^r))$. Taking the $1/r$ multiple of the restriction of Fubini-Study metric of $\mathbf{P}(H^2(M, K^r))$, one has a sequence of metrics on M. One would like to know if, as r tends to infinity, a limiting metric exists. If such a metric does exist, it should be "canonical" and hopefully Kähler-Einstein.

For a complex manifold Ω there are two other intrinsically defined pseudometrics: the Kobayashi-Royden metric and the Caratheodory metric. Let Δ be the Poincaré disk in $\mathbf{C}$. We denote by $\Delta(\Omega)$ the set of holomorphic maps from Ω to Δ, $\Omega(\Delta)$ the set of holomorphic maps from Δ to Ω. Fix the Poincaré distance on Δ. The Caratheodory metric is defined by

$$F_\Omega\colon T\Omega \to \mathbf{R}^+ \quad \text{where} \quad F_\Omega(z, z) = \sup\{|f_*(z)| : f \in \Delta(\Omega), f(z) = 0\}.$$

The Kobayashi-Royden metric on Ω is defined by

$$F_k\colon T\Omega \to \mathbf{R}^+ \quad \text{where} \quad F_k(z, \xi) = \inf\{|u| : f \in \Omega(\Delta), f(0) = z, f_*(u) = \xi\}.$$

Clearly, these two intrinsically defined metrics are distance decreasing under holomorphic maps and invariant under biholomorphic maps.

B. Wong [Wo1] has shown that the holomorphic sectional curvature of the Caratheodory metric is less than or equal to -4, whereas the holomorphic sectional curvature of the Kobayashi metric is not less than -4 when the metric is nontrivial (for the Bergman metric, it is known that the holomorphic sectional curvature is not greater than 4). However, one disadvantage of these two metrics is that they are neither bilinear nor smooth on the tangent spaces (F is only upper-semicontinuous in general).

In some special cases we have a better understanding of these two metrics. For example, a manifold with strongly negative holomorphic sectional curvature always admits a nontrivial Kobayashi-Royden metric. The major theorem in this subject is due to Royden who showed that the Kobayashi-Royden metric is actually the Teichmüller metric. It is a curious fact that the Teichmüller metric has constant holomorphic sectional curvature. Can we classify those complex manifolds that admit Finsler metric with constant holomorphic sectional curvature?

Lempert [Le1], [Le2] proved that the Kobayashi and Caratheodory metrics are actually the same for convex domains in $\mathbf{C}^n$. By using the existence of an extremal mapping, he constructed a lot of bounded holomorphic functions. His theory only works for convex domains; still, it is interesting to see how one can generalize his ideas or use these two metrics to construct bounded holomorphic functions on more general manifolds.

Another interesting fact, proved by B. Wong [Wo2], is that if a smooth, bounded domain in $\mathbf{C}^n$ covers a closed manifold, then it must be the unit ball. This partially confirms the conjecture that a bounded convex domain (not required to be smooth) which covers a closed manifold must be symmetric. His proof needed the boundary estimate of the Kobayashi and Caratheodory metrics.

In general, one would like to compare the Bergman, Kobayashi-Royden, Caratheodory metrics and the Kähler-Einstein metric discussed in the next section. We know that the Caratheodory metric is the smallest of the three. This can be seen by using the generalized Schwarz lemma for Kähler manifolds [Y4]. Yau (see the later improvement by Chan-Cheng-Lu) proved that if $f: M \to N$ is a holomorphic map where M is a complete Kähler manifold with Ricci curvature bounded from below by a constant and N is a Hermitian manifold with holomorphic sectional curvature bounded from above by a negative constant, then f decreases distances up to a constant depending on the curvatures of M and N. Is this true if N is only a Finsler space? If it were true, then one expects that Teichmüller metric is uniformly equivalent to the Kähler-Einstein metric.

2. Kähler-Einstein Metrics on Compact Kähler Manifolds

Let M be a compact Kähler manifold. A necessary condition for the existence of a Kähler-Einstein metric on M is as follows.

(*) There exists a Kähler class Ω such that the first Chern class $c_1(M)$ is cohomologous to some real constant multiple of Ω.

This condition is equivalent to the following:

(*)′ The first Chern class satisfies $c_1(M) > 0$, $c_1(M) = 0$ or $c_1(M) < 0$.

It was proved by the author [Y1], [Y2] that when $c_1(M) = 0$ or $c_1(M) < 0$, (for the latter case see also Aubin [Au3]) there exists in every Kähler class a unique Kähler-Einstein metric. When $c_1(M) > 0$, the space Kähler-Einstein metrics are invariant under automorphism group. However, existence does not hold in general and one would like to impose conditions on M to ensure existence.

We now consider the obstruction, due to Futaki [Fu1], to the existence of Kähler-Einstein metrics when $c_1(M) > 0$; we also consider the notion of "extremal metrics" due to Calabi [Ca2]. Fix a Kähler class $\Omega = [\omega] \in H^{1,1}(M)$ on a compact Kähler manifold M and denote by H_Ω the space of all Kähler metrics with Kähler class Ω. Define the functional

$$F: H_\Omega \to \mathbf{R} \quad \text{by} \quad F:(g) \to \int_M R^2,$$

where R denotes the scalar curvature of the metric g. Calabi called a critical point of this functional an extremal metric. Any Kähler-Einstein metric