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where the degree of  $g_i$  is  $x_i - x_{i-1}$  and all the roots of  $g_i(x)$  in  $\bar{\mathbf{Q}}_p$  have valuation  $-\left(\frac{y_i - y_{i-1}}{x_i - x_{i-1}}\right)$ .

We call the rational numbers,  $\frac{y_i - y_{i-1}}{x_i - x_{i-1}}$ , the slopes of  $g$ .

*Example.* The polynomial  $f_7$  has three factors over  $\mathbf{Q}_2$ , of degrees 4, 2 and 1, respectively, which have slopes  $-3/4$ ,  $-1/2$  and 0.

**COROLLARY.** Let  $d$  be a positive integer. Suppose that  $d$  divides the denominator of each slope (in lowest terms) of  $g$ . Then  $d$  divides the degree of each factor of  $g$  over  $\mathbf{Q}_p$ .

*Proof.* It suffices to show that  $d$  divides the degree of each irreducible factor of  $g$ . Let  $h$  be such a factor. Let  $\alpha \in \bar{\mathbf{Q}}_p$  be a root of  $h$ . Since  $d$  divides the denominator of the valuation of  $\alpha$  (by Theorem NP), it follows that  $d$  divides the index of ramification of the extension  $\mathbf{Q}_p(\alpha)/\mathbf{Q}_p$  which divides the degree of the extension which equals the degree of  $h$ .

## II. APPLICATION TO THE EXPONENTIAL TAYLOR POLYNOMIALS

Fix a prime number  $p$ .

**LEMMA.** Suppose  $k$  is a positive integer and

$$k = a_0 + a_1p + \dots + a_s p^s$$

where  $0 \leq a_i < p$ . Then

$$\text{ord}(k!) = \frac{k - (a_0 + a_1 + \dots + a_s)}{p - 1}.$$

This is easy and well known.

Now write

$$n = b_1 p^{n_1} + b_2 p^{n_2} + \dots + b_s p^{n_s}$$

where  $n_1 > n_2 > \dots > n_s$  and  $0 < b_i < p$ . Let

$$x_i = b_1 p^{n_1} + \dots + b_i p^{n_i}.$$

LEMMA. *The vertices of the Newton polygon of  $f_n$  are*

$$(x_i, -\text{ord}_p(x_i!)), \quad 1 < i < s.$$

This follows easily from the previous lemmas.

It follows that the slopes of  $f$  are

$$m = \frac{-\text{ord}_p(x_i!) + \text{ord}_p(x_{i-1}!)}{x_i - x_{i-1}} = \frac{-(p^{n_i} - 1)}{p^{n_i}(p - 1)}.$$

COROLLARY A. *Suppose that  $p^m$  divides  $n$ . Then  $p^m$  divides the degree of each factor of  $f_n$  over  $\mathbf{Q}_p$ .*

*Proof.* Since  $p^m$  divides  $n$ ,  $m \leq n_s < n_{s-1} < \dots$ . Hence, it follows from (1) that  $p^m$  divides the denominator of each  $m$ . Therefore the corollary follows from the corollary to Theorem NP.

COROLLARY B. *Suppose that  $p^k \leq n$ . Then  $p^k$  divides the degree of the splitting field of  $f_n$  over  $\mathbf{Q}_p$ .*

*Proof.* The hypotheses imply that  $k \leq n_1$ . Hence  $p^k$  divides the denominator of  $m_1$ . As above this implies that  $p^k$  divides the degree of any extension of  $\mathbf{Q}_p$  formed by adjoining a root of  $f_n$  with valuation  $-m_1$ . This yields the corollary.

### III. GLOBAL CONCLUSIONS A AND B

A.  $f_n$  is irreducible.

Suppose

$$n = \prod_p p^{n_p}$$

is the prime factorization of  $n$ . Corollary A implies that, for each prime  $p$ ,  $p^{n_p}$  divides the degree of each factor of  $f_n$  over  $\mathbf{Q}$ . The conclusion follows.

B. Suppose  $n/2 < p \leq n$  is a prime number. Then  $G$  contains a  $p$ -cycle.

By Corollary B,  $p$  divides the degree of the splitting field of  $f_n$  over  $\mathbf{Q}_p$  which divides the degree of the splitting field of  $f_n$  over  $\mathbf{Q}$ . Hence  $p$  divides the order of  $G_n$ . By Cauchy's Theorem  $G$  contains an element of order  $p$ . The conclusion follows since the only elements of order  $p$  in  $S_n$  are  $p$ -cycles if  $p > n/2$ .