GEODESICS IN THE UNIT TANGENT BUNDLE OF A ROUND SPHERE

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What is the optimal unit vector field that can be drawn on the round n-sphere S^n? If we interpret optimality to mean that the vector field has minimum volume when viewed as a cross-section of the sphere's unit tangent bundle US^n, then it is known [Gl-Zi] that a unit vector field on the 3-sphere is optimal if and only if it is tangent to a Hopf fibration. But it is also known [Jo] that these Hopf vector fields are no longer optimal on the 5-sphere. Sharon Pedersen has recently discovered [Pe] that on spheres of dimension at least 5, there are unit vector fields of exceptionally small volume, converging to a vector field with one singularity. Her results suggest the possibility that, beginning on the 5-sphere, there are no vector fields of minimum volume. Her methods show that an understanding of the geometry of a sphere's unit tangent bundle can be expected to play a central role in future investigations in these directions. Inspired by her results, we give here a completely elementary and self-contained determination of the geodesics in the unit tangent bundle US^n.

Let $(p(t), v(t))$ be a constant speed geodesic in US^n, with its usual metric (which we will describe in the next section). We will quickly learn the following:

1) The foot point $p(t)$ need not travel along a geodesic in S^n as it would in the flat case of R^n. But it does trace out a spherical helix, lying entirely within some great 3-sphere S^3 in S^n, and it does so at constant speed $|p'(t)|$.

2) The vector $v(t)$ is tangent to S^3 at the point $p(t)$, and has constant coefficients with respect to the usual tangent-normal-binormal Frenet frame of the helix $p(t)$:

$$v(t) = aT(t) + bN(t) + cB(t) .$$

In particular, it too has constant speed $|v'(t)|$, meaning the norm of its covariant derivative is constant.

A quick dimension count shows that not every curve $(p(t), v(t))$ in the unit tangent bundle US^n which satisfies the above conditions can be a geodesic: there must be one further constraint. To express this, we define two quantities, as follows.

Given such a curve $(p(t), v(t))$, by its *slope* we will mean the ratio of its "vertical" speed $|v'(t)|$ to its "horizontal" speed $|p'(t)|$. Since these are both constant, so is the slope.

We define the *writhe* of the helix $p(t)$ to be

$$\sqrt{(\text{curvature})^2 + (\text{torsion})^2}\,.$$

It too is constant.

Fundamental Constraint. *The curve* $(p(t), v(t))$ *in* US^n, *satisfying 1) and 2) above, is a geodesic there if and only if*

$$SLOPE = WRITHE\,.$$

If $p(t)$ is constant, then neither "slope" nor "writhe" are defined. If $p(t)$ is a great circle in S^n, then we take its "torsion", and hence its "writhe", to be undefined. In each of these cases, we set the interpretation of the Fundamental Constraint as follows.

If $p(t)$ is a constant point, then $(p(t), v(t))$ will be a geodesic in US^n if and only if $v(t)$ traces out a great circle in the tangent space to S^n at that point.

If $p(t)$ is a great circle in S^n, travelled at constant speed, then $(p(t), v(t))$ will be a geodesic in US^n if and only if $v(t)$ spins at constant but arbitary speed along a great circle orthogonal to that of $p(t)$. If this speed is zero, then $v(t)$ is a parallel vector field along $p(t)$.

We can re-interpret the Fundamental Constraint, as follows. Let $p(t)$ be a spherical helix, travelled at unit speed, inside some great 3-sphere S^3 in S^n. Consider the Frenet frame $T(t)$, $N(t)$, $B(t)$ along $p(t)$, and the Frenet equations:

$$T' = \kappa N\,, \quad N' = -\kappa T - \tau B\,, \quad B' = \tau N\,,$$

where κ = curvature and τ = torsion. The vector $U = \tau T - \kappa B$ satisfies $U' = 0$. We call U the *instantaneous axis vector* of our helix. It spans the unique direction along $p(t)$ which appears constant in both Frenet and parallel frames.

Fundamental Constraint (2nd version). *Let* $p(t)$ *be a spherical helix lying in a great 3-sphere inside* S^n, *and let* $v(t)$ *be a unit vector field along* $p(t)$ *which appears constant in Frenet coordinates. Then* $(p(t), v(t))$ *is a geodesic in* US^n *if and only if* $v(t)$ *is orthogonal to the instantaneous axis of the helix.*

The paper is organized into the following sections:

1. *Geometry of the unit tangent bundle.* We describe the metric in two ways, and when the base space is a round sphere, we see that geodesics in its unit tangent bundle project to spherical helices on the sphere.

2. *Geodesics in US^2.* Some of the phenomena show up in this case.

3. *Helices in S^3.* Frenet equations, curvature, torsion and writhe.

4. *Sasaki's equations.* A general calculus for geodesics in the unit tangent bundle UM of any Riemannian manifold M.

5. *Proof of the Fundamental Constraint.* A blend of the Sasaki and Frenet equations.

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1. Geometry of the unit tangent bundle

Let M be an n-dimensional Riemannian manifold, and $(p(t), v(t))$ a path in its unit tangent bundle UM. It is customary to give UM the Riemannian metric in which arc length $s(t)$ along this path is given by the formula

$$s'(t)^2 = |p'(t)|^2 + |v'(t)|^2,$$

where

$$\begin{aligned} p'(t) &= \text{tangent vector to the curve } p(t) \text{ in } M, \\ v'(t) &= \text{covariant derivative of } v(t) \text{ along } p(t) \text{ in } M, \end{aligned}$$

and the norms of these vectors are measured in the given Riemannian metric on M.

When M is flat, and hence parallel translation is independent of path, the above metric on UM is simply the product metric of $M \times S^{n-1}$. So the constant speed geodesics in UM, for example, are just the paths $(p(t), v(t))$ for which $p(t)$ and $v(t)$ are themselves constant speed geodesics in their respective spaces. In particular, each geodesic in UM certainly projects to a geodesic in M.

But when M is curved, the story is quite different. A geodesic in the unit tangent bundle UM need not project to a geodesic in M. We can already see this when M is a round two-sphere.

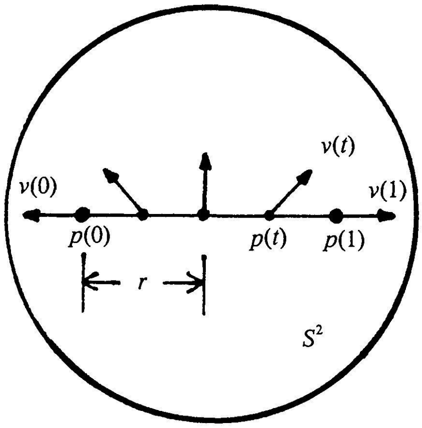

FIGURE 1

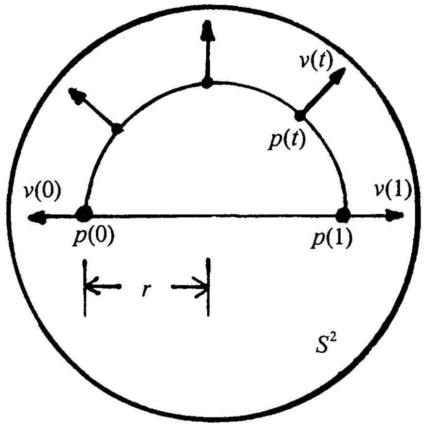

FIGURE 2

In each of Figures 1 and 2, we depict a path $(p(t), v(t))$ in the unit tangent bundle US^2 of a round two-sphere S^2 of radius 1. Though the paths are different, their initial points are the same and their terminal points are the same.

In the first path, the point $p(t)$ travels at constant speed along a geodesic of length $2r$ on S^2. At the same time the tangent vector $v(t)$ rotates at constant speed with respect to a parallel coordinate frame, turning through a total angle π from beginning to end. The length of this path $(p(t), v(t))$ is

$$\sqrt{\pi^2 + 4r^2}\,.$$

If the base space were R^2 instead of S^2, this path in the unit tangent bundle would be a geodesic, indeed a shortest connection between its endpoints.

In the second path, the point $p(t)$ travels at constant speed along a semicircle of length $\pi \sin r$. At the same time the tangent vector $v(t)$ rotates at constant speed with respect to a parallel coordinate frame, turning through a total angle somewhat less than π because of the curvature in the base space S^2. The savings is half of the area $2\pi(1-\cos r)$ inside the small circle. Hence the total angle that $v(t)$ turns through is $\pi \cos r$. It follows that the length of this second path $(p(t), v(t))$ is π.

So the second path is shorter than the first. Indeed, it is a minimizing geodesic in US^2 between its endpoints, whose distance apart is therefore π.

Yet its projection on the base space S^2 is a small circle, not a geodesic.

Immediately one asks: *which curves on* S^n *are projections of geodesics in* US^n*?*

In answering this, we use another approach to the geometry of US^n, viewing it as the homogeneous space $SO(n+1)/SO(n-1)$. Here, the special orthogonal group $SO(n+1)$ is given the usual bi-invariant Riemannian metric, and then the inner products in directions orthogonal to the cosets of $SO(n-1)$ are transfered to the coset space $SO(n+1)/SO(n-1)$. This makes the projection map from $SO(n+1)$ to US^n a Riemannian submersion. We leave it as an exercise to show that this Riemannian metric on US^n coincides with the one described earlier.

A geodesic in $SO(n+1)$ which starts out orthogonal to one of the cosets of $SO(n-1)$ remains orthogonal to all the cosets, and projects to a geodesic in $SO(n+1)/SO(n-1) = US^n$. Furthermore, all the geodesics in US^n are obtained this way.

Suppose, for example, that $n = 3$. If $\big(p(t), v(t)\big)$ is a geodesic in US^3, then by the above, there must be a geodesic $h(t)$ through the identity in $SO(4)$ such that

$$h(t)\big(p(0)\big) = p(t) \quad \text{and} \quad h(t)\big(v(0)\big) = v(t)\,.$$

But every such geodesic $h(t)$ in $SO(4)$ consists of independent, constant speed rotations in a pair of orthogonal two-planes in four-space. Hence $p(t)$ travels along a spiral on an invariant torus, that is, along a spherical helix.

Notice that the isometry $h(t)$ which takes $p(0)$ to $p(t)$ and $v(0)$ to $v(t)$, also takes the entire helix $\{p(t)\}$ to itself. Hence it takes the Frenet frame of the helix at $p(0)$ to the Frenet frame at $p(t)$. It follows that

$$v(t) = aT(t) + bN(t) + cB(t)$$

has constant coeffients with respect to this Frenet frame.

Beyond S^3, nothing new happens for geodesics: it is easy to see that every geodesic in US^n lies inside a totally geodesic submanifold US^3. Indeed, if (p, v) and (q, w) are nearby points on the geodesic, then the vectors p, v, q and w determine the corresponding S^3.

When it comes to proving the Fundamental Constraint, we will capitalize on this observation by restricting our attention to S^3.

We conclude: *the only curves on* S^n *which can be projections of geodesics on* US^n *are spherical helixes (allowing great and small circles and points as special cases) which lie on great 3-spheres. All such spherical helixes will appear in this way.*

2. Geodesics in US^2

If $(p(t), v(t))$ is a geodesic in the unit tangent bundle US^2, then by the discussion in the preceding section, there must be a geodesic $h(t)$ through the identity in $SO(3)$ such that

$$h(t)\,(p(0)) = p(t) \quad \text{and} \quad h(t)\,(v(0)) = v(t)\,.$$

But $h(t)$ must fix a line in three-space, and rotate the orthogonal two-plane at constant speed. Hence $p(t)$, if it moves at all, must travel along a great or small circle, and $v(t)$ must make a constant angle with this circle.

A concrete distance formula between points (p, v) and (q, w) in US^2 is easily obtained. Let δ denote the distance between p and q on S^2, with $0 \leqslant \delta \leqslant \pi$. If this distance is less than π, that is, if p and q are not antipodal, then parallel translate v along the smaller arc of the unique great circle between p and q, and let ε denote the angle at q between this parallel translate of v and the vector w, as shown in Figure 3. If $\delta = \pi$, set $\varepsilon = 0$. Finally, let d denote the distance between (p, v) and (q, w) in US^2. Then a straightforward calculation reveals the formula

$$\cos(d/2) = \cos(\delta/2)\cos(\varepsilon/2)\,,$$

which is just the Pythagorean formula on a round sphere of radius 2, as indicated in Figure 4. Indeed, we have

$$US^2 = SO(3)/SO(1) = SO(3)\,,$$

a round, real projective 3-space.

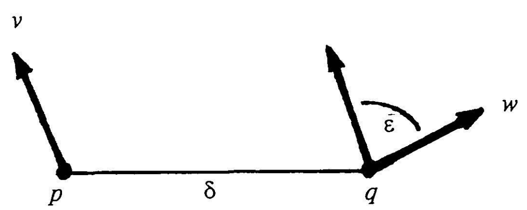

Figure 3

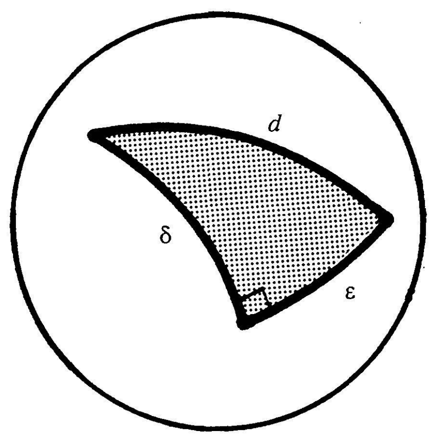

Figure 4

3. Helices in S^3

A spherical helix in S^3 is a curve $p(t)$ of constant geodesic curvature and torsion. As in R^3, two spherical helices of the same curvature and torsion are congruent.

If the curvature is nonzero, then we can define a Frenet frame $T(t)$, $N(t)$, $B(t)$ along $p(t)$ in the usual way, and get the Frenet equations:

$$T' = \kappa N \,, \quad N' = -\kappa T - \tau B \,, \quad B' = \tau N \,.$$

Here we assume that t is an arc length parameter along $p(t)$, and use primes $'$ to denote covariant differentiation of vector fields along this path.

A model helix in S^3 is given by

$$p(t) = (\cos \alpha \cos at, \ \cos \alpha \sin at, \ \sin \alpha \cos bt, \ \sin \alpha \sin bt) \,.$$

Here α ranges between 0 and $\pi/2$ and determines the shape of the flat torus

$$x_1^2 + x_2^2 = \cos^2 \alpha \,, \quad x_3^2 + x_4^2 = \sin^2 \alpha \,,$$

on which the helix $p(t)$ lies. We take the numbers a and b to be $\geqslant 0$, and require that

$$a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = 1 \,,$$

so that the helix will be traversed at unit speed. Every spherical helix in S^3 is congruent to one of these models.

Next, we give formulas for the curvature κ, torsion τ, and writhe $\rho = \sqrt{\kappa^2 + \tau^2}$ of the model helix $p(t)$ in terms of the descriptive parameters α, a and b. These formulas are given as general information only, and will not be used here.

We first record two simple inequalities which follow from the equality $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = 1$.

Note that $a = 1$ and $b = 1$ satisfies this equation. So if one of these quantities increases above 1, the other must decrease below 1. Arranging matters so that a is the larger of the two, we will then have

$$(a^2 - 1)(1 - b^2) \geqslant 0 \,.$$

In addition,

$$a^2 + b^2 \geqslant a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = 1 \,,$$

so we have

$$a^2 + b^2 - 1 \geqslant 0 \,.$$

The formulas for curvature, torsion and writhe are as follows.

$$\begin{aligned}\text{Curvature} &= \kappa = \sqrt{(a^2-1)(1-b^2)}\\ \text{Torsion} &= \tau = ab\\ \text{Writhe} &= \rho = \sqrt{a^2+b^2-1}\,.\end{aligned}$$

Consider the 3-dimensional linear space of vector fields

$$aT(t) + bN(t) + cB(t)$$

which can be written as constant coefficient combinations of the Frenet vectors along the helix $p(t)$. Covariant differentiation along the helix maps this linear space to itself according to the Frenet formulas.

We've already noted in the introduction that the instantaneous axis vector $U = \tau T - \kappa B$ satisfies $U' = 0$.

Consider the vectors N and $V = (\kappa/\rho)T + (\tau/\rho)B$, which form an orthonormal basis for the orthogonal complement of U. Note that

$$\begin{aligned}N' &= -\kappa T - \tau B = -\rho V\,, \quad \text{and}\\ V' &= (\kappa/\rho)T' + (\tau/\rho)B' = (\kappa/\rho)(\kappa N) + (\tau/\rho)(\tau N) = \rho N\,.\end{aligned}$$

Thus, covariant differentiation along the helix kills the instantaneous axis vector and takes the orthogonal 2-plane to itself by a 90° rotation, followed by multiplication by the writhe.

4. Sasaki's equations

Let M be any Riemannian manifold, and UM its unit tangent bundle with the Riemannian metric described in section 1.

THEOREM (Sasaki [Sa], 1958). *The curve $(p(t), v(t))$ in UM is a constant speed geodesic there if and only if both of the following equations hold:*

1) $v'' = -\langle v', v'\rangle\, v$
2) $p'' = R(v', v)p'$.

Here, primes denote ordinary derivatives with respect to t when applied to functions, and covariant derivatives along the path $p(t)$ when applied to vector fields. For example, the first prime in p'' represents ordinary differentiation, the second, covariant differentiation. The symbol R denotes the Riemann curvature transformation

$$R\colon TM_p \times TM_p \to \mathrm{Hom}(TM_p, TM_p)\,.$$

We give a quick proof of Sasaki's theorem, and refer the reader interested in further details both to Sasaki's original paper and to a brief treatment of his result in [Ba-Br-Bu, pages 37-39].

First note that the energy of the curve $(p(t), v(t))$ in UM is given by

$$E = 1/2 \int_0^1 <p', p'> dt + 1/2 \int_0^1 <v', v'> dt .$$

This curve is a geodesic in UM precisely when it is a critical point of E for fixed end point variations. These include variations which fix all the foot points $p(t)$, that is, fixed end point variations of the second integral. This second integral equals the energy of the curve $u(t)$, lying in the unit sphere in the tangent space to M at $p(0)$, obtained by parallel translating $v(t)$ backwards along $p(t)$ to $p(0)$. Hence the curve $u(t)$ is a geodesic, that is, a great circle arc, in this unit sphere.

Because $u(t)$ is a unit vector field, $<u, u> = 1$. Differentiating twice, $<u'', u> + <u', u'> = 0$. Because $u(t)$ runs at constant speed along a great circle, u'' is parallel to u. Hence $u'' = - <u', u'> u$. Parallel translating this equation back out along $p(t)$, we get Sasaki's first equation.

To get Sasaki's second equation, consider a fixed end point variation $(p(t, s), v(t, s))$ of the curve $(p(t), v(t))$ in UM. Suppose this curve is a critical point of the energy E for such variations. Then

$$0 = dE/ds = 1/2 \int_0^1 \partial/\partial s <p', p'> dt + 1/2 \int_0^1 \partial/\partial s <v', v'> dt .$$

The first integrand is processed by differentiating with respect to s, then interchanging the order of the t and s differentiations, and finally setting up for integration by parts, yielding

$$\partial/\partial t <\partial p/\partial s, p'> - <\partial p/\partial s, p''> .$$

The second integrand is processed similarly, except that the Riemann curvature transformation appears as a penalty for interchanging the order of the t and s differentiations, since this time both are covariant. We get

$$\partial/\partial t <\partial v/\partial s, v'> - <\partial v/\partial s, v''> + <R(\partial p/\partial s, p')v, v'> .$$

Integrating these two expressions with respect to t, as required, the leading term of each drops out because the variation is fixed end point. Furthermore, the second term of the second expression is dead zero: since $<v, v> = 1$,

$\partial v/\partial s$ is orthogonal to v, while by Sasaki's first equation, v'' is parallel to v. We get

$$0 = \int_0^1 <\partial p/\partial s, p''> - <R(\partial p/\partial s, p')v, v'> dt .$$

Capitalizing on the symmetries of the curvature, we rewrite this as

$$0 = \int_0^1 <p'' - R(v', v)p', \partial p/\partial s> dt .$$

Since $p(t, s)$ was an arbitrary fixed end point variation, we get

$$p'' - R(v', v)p' = 0 ,$$

which is Sasaki's second equation.

Thus if the curve $(p(t), v(t))$ is a geodesic in UM, then both of Sasaki's equations must be satisfied. Conversely, if these equations are satisfied, then the curve is a critical point of the energy E for fixed end point variations, and hence a geodesic in UM. This completes the proof of Sasaki's theorem.

Here are some immediate consequences of Sasaki's theorem.

Suppose $(p(t), v(t))$ is a constant speed geodesic in UM. Then:

1) The vertical speed $| v'(t) |$ is constant. Indeed,

$$<v, v> = 1 \Rightarrow <v, v'> = 0 ,$$

and hence

$$\partial/\partial t <v', v'> = 2 <v'', v'> = - 2 <v', v'> <v, v'> = 0 ,$$

by Sasaki's first equation.

2) The horizontal speed $| p'(t) |$ is also constant. We have

$$\partial/\partial t <p', p'> = 2 <p'', p'> = 2 <R(v', v)p', p'> = 0 ,$$

by Sasaki's second equation together with the skew-symmetry of the Riemann curvature tensor $<R(\cdot,\cdot)\cdot,\cdot>$ in its last two positions.

3) If $v(t)$ is a parallel vector field along $p(t)$, then Sasaki's second equation reduces to the equation $p'' = 0$ of a geodesic in M. Conversely, if $p(t)$ is a geodesic in M and $v(t)$ a parallel unit vector field along it, then Sasaki's two equations are clearly satisfied, so $(p(t), v(t))$ must be a geodesic in UM. But there will also be geodesics $(p(t), v(t))$ in UM for which $p(t)$ is a geodesic in M, while $v(t)$ is *not* parallel along $p(t)$.

5. Proof of the Fundamental Constraint

Let $(p(t), v(t))$ be a curve in the unit tangent bundle US^3, such that $p(t)$ traces out a spherical helix in S^3 at constant speed, while $v(t)$ has constant coefficients with respect to the moving Frenet frame along this helix. We saw in section 1 that a geodesic in the unit tangent bundle must have this form, and also noted there that it will be sufficient to restrict our attention to the 3-sphere S^3.

In this section we will verify the Fundamental Constraint: $(p(t), v(t))$ is a geodesic in US^3 if and only if its slope equals the writhe of the helix $p(t)$. We will assume that the helix has nonzero curvature, and leave the degenerate case, in which $p(t)$ is a point or a great circle, until the very end.

The key step in the argument may be described as follows. Consider the 3-dimensional linear space of vector fields $aT(t) + bN(t) + cB(t)$ which can be written as constant coefficient combinations of the Frenet vectors along the helix $p(t)$. Covariant differentiation along the helix provides an endomorphism of this space, whose action was described in section 3. If we fix the value of t, this space becomes the tangent space to S^3 at $p(t)$. Here we may consider the action of the Riemann curvature transformation $R(v', v)$. The key step will be to compare these two endomorphisms.

In carrying out the argument, we will be blending Sasaki's two equations:

1) $v'' = - \langle v', v' \rangle v$

2) $p'' = R(v', v)p'$

with the three Frenet equations for the helix:

3) $T' = \kappa N$

4) $N' = -\kappa T - \tau B$

5) $B' = \tau N$.

To begin, assume that $(p(t), v(t))$ is a geodesic in US^3. For convenience, let t be an arc length parameter along $p(t)$. We first aim to show that the action of covariant differentiation coincides with that of the Riemann curvature transformation $R(v', v)$. To do this, we must verify

6) $T' = R(v', v)T$

7) $N' = R(v', v)N$

8) $B' = R(v', v)B$.

The unit tangent vector field $T(t) = p'(t)$, since t was set as an arc length parameter along $p(t)$. Making this substitution in Sasaki's equation 2) gives equation 6).

To get equation 7), combine equations 3) and 6) to get

9) $\kappa N = R(v', v)T$.

Then take covariant derivatives on both sides of this equation:

$$\kappa N' = R(v'', v)T + R(v', v')T + R(v', v)T' .$$

Sasaki's equation 1) and skew symmetry of R show that $R(v'', v) = 0$. Skew-symmetry alone gives $R(v', v') = 0$. In the third term on the right, replace T' by κN. Divide through by κ to get equation 7).

Covariant differentiation and the Riemann curvature transformation $R(v', v)$ are both skew symmetric endomorphisms of our 3-dimensional linear space. Equations 6) and 7) tell us that they agree on two out of the three basis vectors. Automatically, they must agree on the third, giving equation 8). Thus the two endomorphisms coincide.

From this, we want to conclude that slope $=$ writhe.

We've already described the action of covariant differentiation in section 3: it kills the instantaneous axis vector $U = \tau T - \kappa B$ and takes the orthogonal 2-plane to itself by a 90° rotation, followed by multiplication by the writhe.

Since we are on S^3, one can show that the Riemann curvature transformation $R(v', v)$ consists of orthogonal projection of the tangent 3-space onto the 2-plane spanned by v and v', followed by rotation by 90° in the direction from v to v', followed by multiplication by $| v' |$.

Since these two transformations coincide, writhe $= | v' |$. All this assumes that $| p' | = 1$. In general, we get

$$\text{writhe} = | v' | / | p' | = \text{slope} ,$$

verifying the necessity of the Fundamental Constraint.

Note also that, because the two transformations coincide, the vector $v(t)$ must be orthogonal to the instantaneous axis vector $U(t)$ of the helix $p(t)$, thus verifying the necessity of the Fundamental Constraint in its second formulation.

Conversely, suppose $(p(t), v(t))$ is a curve in US^3, with $p(t)$ tracing out a spherical helix in S^3 at constant speed, and $v(t)$ having constant coefficients with respect to the moving Frenet frame along this helix. In particular, $| v'(t) |$ is constant, and hence so is the slope $| v'(t) | / | p'(t) |$. Suppose this slope equals the writhe of the helix. We must show that $(p(t), v(t))$ is a geodesic in US^3.

As in the first part of the proof, we aim to show that the action of covariant differentiation coincides with that of the Riemann curvature transformation $R(v', v)$.

To this end, adjust the speed so that t is an arc length parameter along the helix $p(t)$. Hence $| v' | =$ writhe. But this is the maximum magnification of covariant differentiation, and can only be achieved when $v(t)$ is orthogonal to the instantaneous axis vector $U(t)$. Thus $<v, U> = 0$. Differentiate this equation, keeping in mind that $U' = 0$, and get $<v', U> = 0$. Hence v' is also orthogonal to the instantaneous axis.

But this means that the kernel and image of covariant differentiation coincide with the kernel and image of the Riemann curvature transformation $R(v', v)$. Since writhe $= | v' |$, the maximum magnifications of these two transformations also coincide. Then, by their special nature, so must the transformations themselves.

With this done, we can now check that $(p(t), v(t))$ is a geodesic in US^3 by verifying Sasaki's two equations.

Consider the vector field v''. Since covariant differentiation coincides with application of $R(v', v)$, the vector v'' is obtained from v by twice rotating the vv' plane by 90° and twice multiplying by $| v' |$. That is,

$$v'' = - <v', v'> v \, ,$$

which is Sasaki's first equation.

Next look at the vector field T'. This must be the same as $R(v', v)T$. But $T(t) = p'(t)$ and $T'(t) = p''(t)$, so we get

$$p'' = R(v', v)p' \, ,$$

which is Sasaki's second equation.

Hence $(p(t), v(t))$ must be a geodesic in US^3 by Sasaki's theorem, verifying the sufficiency of the Fundamental Constraint.

To verify the sufficiency of the Fundamental Constraint in its second formulation, suppose we begin instead with the information that $v(t)$ is orthogonal to the instantaneous axis vector $U(t)$. It is here that covariant differentiation achieves its maximum magnification, equal to the writhe of the helix $p(t)$. Thus $| v'(t) | =$ writhe. The above proof of sufficiency now applies, and we conclude again that $(p(t), v(t))$ must be a geodesic in US^3.

We complete the proof of the Fundamental Constraint by checking the two degenerate cases, again using Sasaki's equations.

If $p(t)$ is a constant point, then Sasaki's second equation is certainly satisfied, while the first tells us that $(p(t), v(t))$ is a geodesic in US^3 if and only if $v(t)$ traces out, at constant speed, a great circle in the tangent space to S^3 at that point.

If $p(t)$ is a great circle in S^3, travelled at constant speed, then $p'' = 0$, so Sasaki's second equation reads

$$R(v', v)p' = 0 .$$

This can be satisfied in two ways.

One is that $v' = 0$, so that $v(t)$ is a parallel vector field along $p(t)$. In this case, Sasaki's first equation is automatically satisfied, so $(p(t), v(t))$ must be a geodesic in US^3.

The other way for Sasaki's second equation to be satisfied is that v and v' are both orthogonal to p'. Parallel translate $v(t)$ backwards along $p(t)$ to the vector field $u(t)$ in the tangent space to S^3 at $p(0)$. Then Sasaki's first equation says that $u(t)$ traces out, at constant speed, a great circle orthogonal to $p'(0)$. Equivalently, $v(t)$ spins at constant but arbitrary speed along a great circle orthogonal to that of $p(t)$. In these circumstances, the curve $(p(t), v(t))$ will be a geodesic in US^3.

But these are precisely the interpretations of the Fundamental Constraint which were set in the introduction, and the proof is complete.

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