1. Geometry of the unit tangent bundle

The paper is organized into the following sections:

1. *Geometry of the unit tangent bundle.* We describe the metric in two ways, and when the base space is a round sphere, we see that geodesics in its unit tangent bundle project to spherical helices on the sphere.

2. *Geodesics in US^2.* Some of the phenomena show up in this case.

3. *Helices in S^3.* Frenet equations, curvature, torsion and writhe.

4. *Sasaki's equations.* A general calculus for geodesics in the unit tangent bundle UM of any Riemannian manifold M.

5. *Proof of the Fundamental Constraint.* A blend of the Sasaki and Frenet equations.

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1. Geometry of the unit tangent bundle

Let M be an n-dimensional Riemannian manifold, and $(p(t), v(t))$ a path in its unit tangent bundle UM. It is customary to give UM the Riemannian metric in which arc length $s(t)$ along this path is given by the formula

$$s'(t)^2 = |\, p'(t)\,|^2 + |\, v'(t)\,|^2 ,$$

where

$$\begin{aligned} p'(t) &= \text{tangent vector to the curve } p(t) \text{ in } M\,, \\ v'(t) &= \text{covariant derivative of } v(t) \text{ along } p(t) \text{ in } M\,, \end{aligned}$$

and the norms of these vectors are measured in the given Riemannian metric on M.

When M is flat, and hence parallel translation is independent of path, the above metric on UM is simply the product metric of $M \times S^{n-1}$. So the constant speed geodesics in UM, for example, are just the paths $(p(t), v(t))$ for which $p(t)$ and $v(t)$ are themselves constant speed geodesics in their respective spaces. In particular, each geodesic in UM certainly projects to a geodesic in M.

But when M is curved, the story is quite different. A geodesic in the unit tangent bundle UM need not project to a geodesic in M. We can already see this when M is a round two-sphere.

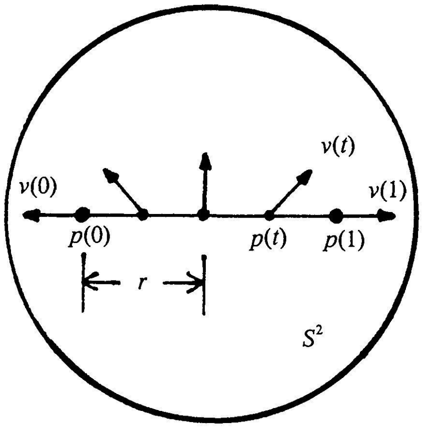

FIGURE 1

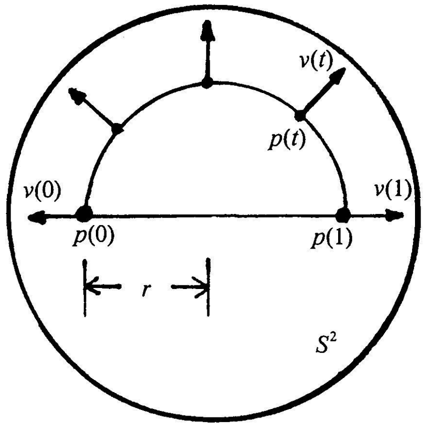

FIGURE 2

In each of Figures 1 and 2, we depict a path $(p(t), v(t))$ in the unit tangent bundle US^2 of a round two-sphere S^2 of radius 1. Though the paths are different, their initial points are the same and their terminal points are the same.

In the first path, the point $p(t)$ travels at constant speed along a geodesic of length $2r$ on S^2. At the same time the tangent vector $v(t)$ rotates at constant speed with respect to a parallel coordinate frame, turning through a total angle π from beginning to end. The length of this path $(p(t), v(t))$ is

$$\sqrt{\pi^2 + 4r^2}\,.$$

If the base space were R^2 instead of S^2, this path in the unit tangent bundle would be a geodesic, indeed a shortest connection between its endpoints.

In the second path, the point $p(t)$ travels at constant speed along a semicircle of length $\pi \sin r$. At the same time the tangent vector $v(t)$ rotates at constant speed with respect to a parallel coordinate frame, turning through a total angle somewhat less than π because of the curvature in the base space S^2. The savings is half of the area $2\pi(1-\cos r)$ inside the small circle. Hence the total angle that $v(t)$ turns through is $\pi \cos r$. It follows that the length of this second path $(p(t), v(t))$ is π.

So the second path is shorter than the first. Indeed, it is a minimizing geodesic in US^2 between its endpoints, whose distance apart is therefore π.

Yet its projection on the base space S^2 is a small circle, not a geodesic.

Immediately one asks: *which curves on* S^n *are projections of geodesics in* US^n*?*

In answering this, we use another approach to the geometry of US^n, viewing it as the homogeneous space $SO(n+1)/SO(n-1)$. Here, the special orthogonal group $SO(n+1)$ is given the usual bi-invariant Riemannian metric, and then the inner products in directions orthogonal to the cosets of $SO(n-1)$ are transfered to the coset space $SO(n+1)/SO(n-1)$. This makes the projection map from $SO(n+1)$ to US^n a Riemannian submersion. We leave it as an exercise to show that this Riemannian metric on US^n coincides with the one described earlier.

A geodesic in $SO(n+1)$ which starts out orthogonal to one of the cosets of $SO(n-1)$ remains orthogonal to all the cosets, and projects to a geodesic in $SO(n+1)/SO(n-1) = US^n$. Furthermore, all the geodesics in US^n are obtained this way.

Suppose, for example, that $n = 3$. If $(p(t), v(t))$ is a geodesic in US^3, then by the above, there must be a geodesic $h(t)$ through the identity in $SO(4)$ such that

$$h(t)\,(p(0)) = p(t) \quad \text{and} \quad h(t)\,(v(0)) = v(t)\,.$$

But every such geodesic $h(t)$ in $SO(4)$ consists of independent, constant speed rotations in a pair of orthogonal two-planes in four-space. Hence $p(t)$ travels along a spiral on an invariant torus, that is, along a spherical helix.

Notice that the isometry $h(t)$ which takes $p(0)$ to $p(t)$ and $v(0)$ to $v(t)$, also takes the entire helix $\{p(t)\}$ to itself. Hence it takes the Frenet frame of the helix at $p(0)$ to the Frenet frame at $p(t)$. It follows that

$$v(t) = aT(t) + bN(t) + cB(t)$$

has constant coeffients with respect to this Frenet frame.

Beyond S^3, nothing new happens for geodesics: it is easy to see that every geodesic in US^n lies inside a totally geodesic submanifold US^3. Indeed, if (p, v) and (q, w) are nearby points on the geodesic, then the vectors p, v, q and w determine the corresponding S^3.

When it comes to proving the Fundamental Constraint, we will capitalize on this observation by restricting our attention to S^3.

We conclude: *the only curves on* S^n *which can be projections of geodesics on* US^n *are spherical helixes (allowing great and small circles and points as special cases) which lie on great 3-spheres. All such spherical helixes will appear in this way.*