

Zeitschrift: L'Enseignement Mathématique
Band: 34 (1988)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE EULER CLASS OF ORTHOGONAL RATIONAL REPRESENTATIONS OF FINITE GROUPS
Kapitel: 1. Invariant Bilinear Forms
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DOI: <https://doi.org/10.5169/seals-56597>

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occasion my gratefulness for his stimulating advices and the interest he constantly showed for this work.

1. INVARIANT BILINEAR FORMS

Let \mathbf{K} be a field of characteristic 0, V a finite dimensional vector space over \mathbf{K} and $\rho: G \rightarrow GL(V)$ a \mathbf{K} -representation of the group G . A \mathbf{K} -bilinearform $\alpha: V \times V \rightarrow \mathbf{K}$ is called ρ -invariant if

$$\alpha(\rho(g)x, \rho(g)y) = \alpha(x, y) \quad \forall x, y \in V, \quad \forall g \in G.$$

If G is finite, then for any bilinear form γ the form $\bar{\gamma}$ defined by

$$\bar{\gamma}(x, y) := \sum_{g \in G} \gamma(\rho(g)x, \rho(g)y)$$

is ρ -invariant.

(1.1) *Remark.* If α is definit (i.e. $\alpha(x, x) = 0 \Rightarrow x = 0$) and if ρ splits in a direct sum $\rho = \rho_1 \oplus \rho_2$, the restriction ρ' of ρ to the orthogonal complement of the invariant space corresponding to ρ_1 is equivalent to ρ_2 . Since we always can substitute a representation or a bilinear form by an equivalent one, we can assume that the representation space of a sum is an orthogonal sum of corresponding invariant subspaces.

We call *standard bilinear form* (of dimension m) the map $\beta_m: \mathbf{K}^m \times \mathbf{K}^m \rightarrow \mathbf{K}$ given by

$$\beta_m(x, y) := \sum_{i=1}^m x_i y_i \quad \text{with} \quad x = (x_1, \dots, x_m) \quad \text{and} \quad y = (y_1, \dots, y_m).$$

The group $O_m(\mathbf{K})$ is the subgroup of $GL_m(\mathbf{K})$ of matrices (a_{ij}) such that $\sum_k a_{ik} a_{jk} = \delta_{ij}$ for all i, j . The group $SO_m(\mathbf{K})$ is the subgroup of $O_m(\mathbf{K})$ of matrices (a_{ij}) with $\det(a_{ij}) = 1$. It is therefore evident that a representation $\rho: G \rightarrow GL_m(\mathbf{K})$ is realizable over $O_m(\mathbf{K})$ if and only if there is a ρ -invariant symmetric bilinear form which is equivalent to the standard bilinear form.

Let p be a prime number. Up to equivalence, there is a unique irreducible faithful \mathbf{Q} -representation σ of \mathbf{Z}/p ; it is given by

$$\begin{aligned} \sigma: \mathbf{Z}/p &\rightarrow GL_{p-1}(\mathbf{Q}) \\ 1 &\mapsto A := \begin{bmatrix} 0 & \cdot & \cdot & \cdot & -1 \\ 1 & \cdot & \cdot & \cdot & -1 \\ & & \cdot & & \\ \cdot & \cdot & \cdot & 1 & -1 \end{bmatrix} \end{aligned}$$

We can identify the irreducible faithful $\mathbf{Q}[\mathbf{Z}/p]$ -Module \mathbf{Q}^{p-1} with $\mathbf{Q}(\zeta_p)$ (ζ_p : primitive p -th root of unity, $1 \in \mathbf{Z}/p$ acts on $\mathbf{Q}(\zeta_p)$ by multiplication with ζ_p). Any symmetric σ -invariant bilinear form is given by $tr_{\mathbf{Q}(\zeta_p)/\mathbf{Q}}(axy\bar{y})$ with $a \in \mathbf{Q}(\zeta_p + \zeta_p^{-1})$ (cf. [4] or [6]). We write γ_a for the σ -invariant bilinear form corresponding to $a \in \mathbf{Q}(\zeta_p + \zeta_p^{-1})$.

(1.2) LEMMA. *The discriminant of γ_a in $\mathbf{Q}/\mathbf{Q}^{*2}$ is equal to $p \bmod \mathbf{Q}^{*2}$.*

Proof. Since $a \in \mathbf{L} := \mathbf{Q}(\zeta_p + \zeta_p^{-1})$ we have: $\gamma_a = tr_{\mathbf{L}/\mathbf{Q}}(tr_{\mathbf{Q}(\zeta_p)/\mathbf{L}}axy\bar{y})$. An easy computation shows that $tr_{\mathbf{Q}(\zeta_p)/\mathbf{L}}(axy\bar{y})$ is a 2-dimensional symmetric \mathbf{L} -bilinearform with discriminant $4 - (\zeta_p + \zeta_p^{-1})^2 \bmod \mathbf{L}^{*2} \in \mathbf{L}/\mathbf{L}^{*2}$. Applying [7, Lemma 2.2] we conclude that the discriminant of γ_a is independant of $a \in \mathbf{L}$. Consider now the matrix representation of σ given before (σ : irreducible faithful \mathbf{Q} -representation of \mathbf{Z}/p). Let C be the $(p-1) \times (p-1)$ -matrix given by:

$$C := \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \cdot & \cdot \\ & & & -1 & 2 \end{bmatrix}$$

It is easy to check that C is the matrix of a σ -invariant symmetric bilinear form. The Lemma follows since the determinant of C is equal to p .

2. ORTHOGONAL REPRESENTATIONS OF p -GROUPS

Let $p > 2$ be an odd prime. The integer $l_{\mathbf{Q}}(p)$ is defined by

$$l_{\mathbf{Q}}(p) := \text{g.c.d.} \left\{ \begin{array}{l} m > 1 \\ \left. \begin{array}{l} \text{the } m\text{-fold direct sum } \sigma \oplus \dots \oplus \sigma \text{ of the irre-} \\ \text{ducible faithful } \mathbf{Q}\text{-representation } \sigma \text{ of } \mathbf{Z}/p \text{ is} \\ \text{equivalent to an orthogonal representation} \end{array} \right\} \end{array} \right\}$$

The importance played by cyclic groups in the investigation of representations of p -groups is given by the following result (cf. [1, Theorem (1.10)]):

(2.1) PROPOSITION. *Let G be a finite p -group ($p > 2$) and let ρ be an irreducible \mathbf{Q} -representation of G . Then either ρ is induced from a representation θ of a normal subgroup of index p , or ρ factors through a \mathbf{Q} -representation of \mathbf{Z}/p .*