**Zeitschrift:** L'Enseignement Mathématique

**Band:** 34 (1988)

**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE EULER CLASS OF ORTHOGONAL RATIONAL REPRESENTATIONS OF FINITE GROUPS

**Kapitel:** 1. Invariant Bilinear Forms

**Autor:** Piveteau, Jean-Marc

**DOI:** https://doi.org/10.5169/seals-56597 occasion my gratefulness for his stimulating advices and the interest he constantly showed for this work.

## 1. Invariant Bilinear Forms

Let $\mathbf{K}$ be a field of characteristic 0, $V$ a finite dimensional vector space over $\mathbf{K}$ and $\rho: G \to GL(V)$ a $\mathbf{K}$-representation of the group $G$. A $\mathbf{K}$-bilinearform $\alpha: V \times V \to \mathbf{K}$ is called $\rho$-*invariant* if

$$\alpha(\rho(g)x, \rho(g)y) = \alpha(x, y) \qquad \forall x, y \in V, \quad \forall g \in G.$$

If $G$ is finite, then for any bilinear form $\gamma$ the form $\bar{\gamma}$ defined by

$$\bar{\gamma}(x, y) := \sum_{g \in G} \gamma(\rho(g)x, \rho(g)y)$$

is $\rho$-invariant.

(1.1) *Remark.* If $\alpha$ is definit (i.e. $\alpha(x, x) = 0 \Rightarrow x = 0$) and if $\rho$ splits in a direct sum $\rho = \rho_1 \oplus \rho_2$, the restriction $\rho'$ of $\rho$ to the orthogonal complement of the invariant space corresponding to $\rho_1$ is equivalent to $\rho_2$. Since we always can substitute a representation or a bilinear form by an equivalent one, we can assume that the representation space of a sum is an orthogonal sum of corresponding invariant subspaces.

We call *standard bilinear form* (of dimension $m$) the map $\beta_m: \mathbf{K}^m \times \mathbf{K}^m \to \mathbf{K}$ given by

$$\beta_m(x, y) := \sum_{i=1}^{m} x_i y_i \quad \text{with} \quad x = (x_1, \dots, x_m) \quad \text{and} \quad y = (y_1, \dots, y_m).$$

The group $O_m(\mathbf{K})$ is the subgroup of $GL_m(\mathbf{K})$ of matrices $(a_{ij})$ such that $\sum_k a_{ik}a_{jk} = \delta_{ij}$ for all $i, j$. The group $SO_m(\mathbf{K})$ is the subgroup of $O_m(\mathbf{K})$ of matrices $(a_{ij})$ with $\det(a_{ij}) = 1$. It is therefore evident that a representation $\rho: G \to GL_m(\mathbf{K})$ is realizable over $O_m(\mathbf{K})$ if and only if there is a $\rho$-invariant symmetric bilinear form which is equivalent to the standard bilinear form.

Let $p$ be a prime number. Up to equivalence, there is a unique irreducible faithful $\mathbf{Q}$-representation $\sigma$ of $\mathbf{Z}/p$; it is given by

$$\sigma: \mathbf{Z}/p \to GL_{p-1}(\mathbf{Q})$$

$$1 \mapsto A := \begin{bmatrix} 0 & . & . & . & -1 \\ 1 & . & . & . & -1 \\ & & & . & \\ . & . & . & 1 & -1 \end{bmatrix}$$

We can identify the irreducible faithful $\mathbf{Q}[\mathbf{Z}/p]$-Module $\mathbf{Q}^{p-1}$ with $\mathbf{Q}(\zeta_p)$ ($\zeta_p$: primitive $p$-th root of unity, $1 \in \mathbf{Z}/p$ acts on $\mathbf{Q}(\zeta_p)$ by multiplication with $\zeta_p$). Any symmetric $\sigma$-invariant bilinear form is given by $tr_{\mathbf{Q}(\zeta_p)/\mathbf{Q}}(ax\bar{y}$ with $a \in \mathbf{Q}(\zeta_p+\zeta_p^{-1})$ (cf. [4] or [6]). We write $\gamma_a$ for the $\sigma$-invariant bilinear form corresponding to $a \in \mathbf{Q}(\zeta_p+\zeta_p^{-1})$.

(1.2) Lemma. *The discriminant of* $\gamma_a$ *in* $\mathbf{Q}/\mathbf{Q}^{*2}$ *is equal to* $p$ *mod* $\mathbf{Q}^{*2}$.

*Proof.* Since $a \in \mathbf{L} := \mathbf{Q}(\zeta_p+\zeta_p^{-1})$ we have: $\gamma_a = tr_{\mathbf{L}/\mathbf{Q}}(tr_{\mathbf{Q}(\zeta_p)/\mathbf{L}}ax\bar{y})$. An easy computation shows that $tr_{\mathbf{Q}(\zeta_p)/\mathbf{L}}(ax\bar{y})$ is a 2-dimensional symmetric $\mathbf{L}$-bilinearform with discriminant $4-(\zeta_p+\zeta_p^{-1})^2$ mod $\mathbf{L}^{*2} \in \mathbf{L}/\mathbf{L}^{*2}$. Applying [7, Lemma 2.2] we conclude that the discriminant of $\gamma_a$ is independant of $a \in \mathbf{L}$. Consider now the matrix representation of $\sigma$ given before ($\sigma$: irreducible faithful $\mathbf{Q}$-representation of $\mathbf{Z}/p$). Let $C$ be the $(p-1)\times(p-1)$-matrix given by:

$$C := \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \cdot & \cdot \\ & & & -1 & 2 \end{bmatrix}$$

It is easy to check that $C$ is the matrix of a $\sigma$-invariant symmetric bilinear form. The Lemma follows since the determinant of $C$ is equal to $p$.

## 2. Orthogonal representations of $p$-groups

Let $p > 2$ be an odd prime. The integer $l_{\mathbf{Q}}(p)$ is defined by

$$l_{\mathbf{Q}}(p) := \text{g.c.d.}\left\{ m > 1 \;\middle|\; \begin{array}{l}\text{the } m\text{-fold direct sum } \sigma \oplus \dots \oplus \sigma \text{ of the irre-}\\ \text{ducible faithful } \mathbf{Q}\text{-representation } \sigma \text{ of } \mathbf{Z}/p \text{ is}\\ \text{equivalent to an orthogonal representation}\end{array}\right\}$$

The importance played by cyclic groups in the investigation of representations of $p$-groups is given by the following result (cf. [1, Theorem (1.10)]):

(2.1) Proposition. *Let* $G$ *be a finite p-group* ($p>2$) *and let* $\rho$ *be an irreducible* $\mathbf{Q}$*-representation of* $G$. *Then either* $\rho$ *is induced from a representation* $\theta$ *of a normal subgroup of index* $p$, *or* $\rho$ *factors through a* $\mathbf{Q}$*-representation of* $\mathbf{Z}/p$.