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Artikel: SOME ALMOST HOMOGENEOUS GROUP ACTIONS ON SMOOTH COMPLETE RATIONAL SURFACES

Kapitel: §3. Application to SL(2)-embeddings

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This finishes Case 3. Thus we know all the embeddings into $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$ and $\mathbf{F}_n$, $n \geqslant 1$. The comments after Theorem 2.1 are easily verified by checking each embedding. This finishes the proof of the theorem. □

Remarks.

(1) Note that — as to be expected — all the embedding into $\mathbf{F}_1$ are obtained by blowing up the embeddings into $\mathbf{P}^2$ at fixed points.

(2) The "exceptional" embeddings, i.e. those with only one fixed point, are of special interest, because this phenomenon does not occur for smooth complete embeddings of tori. (See [KKMS] for a reference on torus embeddings.)

§ 3. APPLICATION TO $SL(2)$-EMBEDDINGS

In [LV] a combinatorical method is presented in order to classify all normal $SL(2)$-embeddings. A natural question is how to classify those which are smooth and complete to obtain a *geometrical* realization. We now sketch how the result of this article is useful for this. (For further details see [JM].)

Given a B/Γ-embedding X, we construct an $SL(2)/\Gamma$-embedding in the following way. Consider the B-action on $SL(2) \times X$ given by

$$b \cdot (s, x) = (sb^{-1}, bx)$$

where $b \in B$, $s \in SL(2)$, and $x \in X$. Denote by $SL(2)*_B X$ the variety obtained by quotienting by this action. The action of $SL(2)$ on this variety by left multiplication endows it with the structure of an $SL(2)/\Gamma$-embedding. The projection $SL(2) \times X \to SL(2)$ induces a locally trivial fibre bundle $SL(2)*_B X \xrightarrow{p} SL(2)/B \cong \mathbf{P}^1$. The morphism p is $SL(2)$-equivariant, and the fibre of p is B-isomorphic to X. So we see that for studying the geometry of the $SL(2)/\Gamma$-embeddings of this form it is useful to study the B/Γ-embeddings.

As for general $SL(2)/\Gamma$-embeddings one finds the following essential result. Let Γ be a finite cyclic subgroup of $SL(2)$. Let V be a smooth $SL(2)/\Gamma$-embedding with orbit Y. Then there exists a Borel subgroup B of $SL(2)$ containing Γ and an $SL(2)$-stable open neighborhood of Y in V which is of the form $SL(2)*_B X$ for some smooth B/Γ-embedding X. Thus all smooth $SL(2)/\Gamma$-embeddings are *locally* of the form above. Also any smooth B/Γ-embedding can be completed to a smooth embedding. Thus it is enough to study the complete ones.

We can use this fact, for example, to study blow-ups of orbits, since blowing up is a local property. Thus we can find the minimal $SL(2)/\Gamma$-embeddings. This is done in [JM], Chapter IV, for $\Gamma = \{ e \}$ and $\Gamma = \{ \pm e \}$.

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