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Conversely, if  $K$  is a field, which is a quadratic extension of  $\mathbf{Q}$ , then it is necessarily of the form  $K = \mathbf{Q}(\sqrt{d})$ , where  $d$  is a square-free integer.

If  $d > 0$  then  $K$  is a subfield of the field  $\mathbf{R}$  of real numbers: it is called a real quadratic field.

If  $d < 0$  then  $K$  is not a subfield of  $\mathbf{R}$ , and it is called an imaginary quadratic field.

If  $\alpha = a + b\sqrt{d} \in K$ , with  $a, b \in \mathbf{Q}$ , its conjugate is  $\alpha' = a - b\sqrt{d}$ . Clearly,  $\alpha = \alpha'$  exactly when  $\alpha \in \mathbf{Q}$ .

The norm of  $\alpha$  is  $N(\alpha) = \alpha\alpha' = a^2 - db^2 \in \mathbf{Q}$ . It is obvious that  $N(\alpha) \neq 0$  exactly when  $\alpha \neq 0$ . If  $\alpha, \beta \in K$  then  $N(\alpha\beta) = N(\alpha)N(\beta)$ ; in particular, if  $\alpha \in \mathbf{Q}$  then  $N(\alpha) = \alpha^2$ .

The trace of  $\alpha$  is  $\text{Tr}(\alpha) = \alpha + \alpha' = 2a \in \mathbf{Q}$ . If  $\alpha, \beta \in K$  then  $\text{Tr}(\alpha + \beta) = \text{Tr}(\alpha) + \text{Tr}(\beta)$ ; in particular, if  $\alpha \in \mathbf{Q}$  then  $\text{Tr}(\alpha) = 2\alpha$ .

It is clear that  $\alpha, \alpha'$  are the roots of the quadratic equation  $X^2 - \text{Tr}(\alpha)X + N(\alpha) = 0$ .

## B) RINGS OF INTEGERS

Let  $K = \mathbf{Q}(\sqrt{d})$ , where  $d$  is a square-free integer.

$\alpha \in K$  is an algebraic integer when there exist integers  $m, n \in \mathbf{Z}$  such that  $\alpha^2 + m\alpha + n = 0$ .

Let  $A$  be the set of all algebraic integers of  $K$ .  $A$  is a subring of  $K$ , which is the field of fractions of  $A$ , and  $A \cap \mathbf{Q} = \mathbf{Z}$ . If  $\alpha \in A$  then the conjugate  $\alpha' \in A$ . Clearly,  $\alpha \in A$  if and only if both  $N(\alpha)$  and  $\text{Tr}(\alpha)$  are in  $\mathbf{Z}$ .

Here is a criterion for the element  $\alpha = a + b\sqrt{d}$  ( $a, b \in \mathbf{Q}$ ) to be an algebraic integer:  $\alpha \in A$  if and only if

$$\begin{cases} 2a = u \in \mathbf{Z}, & 2b = v \in \mathbf{Z} \\ u^2 - dv^2 \equiv 0 \pmod{4}. \end{cases}$$

Using this criterion, it may be shown:

If  $d \equiv 2$  or  $3 \pmod{4}$  then  $A = \{a + b\sqrt{d} \mid a, b \in \mathbf{Z}\}$ .

If  $d \equiv 1 \pmod{4}$  then  $A = \left\{ \frac{a + b\sqrt{d}}{2} \mid a, b \in \mathbf{Z}, a \equiv b \pmod{2} \right\}$ .

If  $\alpha_1, \alpha_2 \in A$  are such that every element  $\alpha \in A$  is uniquely of the form  $\alpha = m_1\alpha_1 + m_2\alpha_2$ , with  $m_1, m_2 \in \mathbf{Z}$ , then  $\{\alpha_1, \alpha_2\}$  is called an integral basis of  $A$ . In other words,  $A = \mathbf{Z}\alpha_1 \oplus \mathbf{Z}\alpha_2$ .

If  $d \equiv 2$  or  $3 \pmod{4}$  then  $\{1, \sqrt{d}\}$  is an integral basis of  $A$ .

If  $d \equiv 1 \pmod{4}$  then  $\left\{1, \frac{1 + \sqrt{d}}{2}\right\}$  is an integral basis of  $A$ .

### C) DISCRIMINANT

Let  $\{\alpha_1, \alpha_2\}$  be an integral basis. Then

$$D = D_K = \det \begin{pmatrix} \text{Tr}(\alpha_1^2) & \text{Tr}(\alpha_1\alpha_2) \\ \text{Tr}(\alpha_1\alpha_2) & \text{Tr}(\alpha_2^2) \end{pmatrix}$$

is independent of the choice of the integral basis. It is called the discriminant of  $K$ . It is a non-zero integer.

If  $d \equiv 2$  or  $3 \pmod{4}$  then

$$D = \det \begin{pmatrix} \text{Tr}(1) & \text{Tr}(\sqrt{d}) \\ \text{Tr}(\sqrt{d}) & \text{Tr}(d) \end{pmatrix} = \det \begin{pmatrix} 2 & 0 \\ 0 & 2d \end{pmatrix} \quad \text{so } D = 4d.$$

If  $d \equiv 1 \pmod{4}$  then

$$D = \det \begin{pmatrix} \text{Tr}(1) & \text{Tr}\left(\frac{1 + \sqrt{d}}{2}\right) \\ \text{Tr}\left(\frac{1 + \sqrt{d}}{2}\right) & \text{Tr}\left(\frac{1 + \sqrt{d}}{2}\right)^2 \end{pmatrix} = \det \begin{pmatrix} 2 & 1 \\ 1 & \frac{1+d}{2} \end{pmatrix} \quad \text{so } D = d.$$

Every discriminant is  $D \equiv 0$  or  $1 \pmod{4}$ .

In terms of the discriminant,

$$A = \left\{ \frac{a + b\sqrt{D}}{2} \mid a, b \in \mathbf{Z}, \quad a^2 \equiv Db^2 \pmod{4} \right\}.$$

### D) DECOMPOSITION OF PRIMES

Let  $K = \mathbf{Q}(\sqrt{d})$ , where  $d$  is a square-free integer, let  $A$  be the ring of integers of  $K$ .

The ideal  $P \neq 0$  of  $A$  is a prime ideal if the residue ring  $A/P$  has no zero-divisors.

If  $P$  is a prime ideal there exists a unique prime number  $p$  such that  $P \cap \mathbf{Z} = \mathbf{Z}p$ , or equivalently,  $P \supseteq Ap$ .