3. Local inversion

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.07.2024**

LEMMA 2.1. *Let* A, B *be metric spaces, with* $A \neq \emptyset$ *and* B *connected. Let* $P: A \to B$ *be a continuous map. Assume:*

(i) P *is open,*

(ii) P *is proper, that is, for any compact subset* K *in* $B, P^{-1}(K)$ *is compact. Then* P *is surjective.*

Proof. We only need to prove that $P(A)$ is closed. Let b_0 be a point in $\overline{P(A)}$. Since B is a metric space, there exists a sequence $(b_i)_{i>0}$ in $P(A)$ converging to b_0. The subset $K = \{b_0, b_1, b_2, ...\}$ is compact, hence so is $PP^{-1}(K)$. The latter contains $b_1, ..., b_i, ...$, hence b_0, and it is obviously contained in $P(A)$. Q.E.D.

In order to make use of this lemma, we shall need some inverse function theorem for (i), and some *a priori* estimates for (ii).

3. LOCAL INVERSION

THEOREM 3.1. *Let* X *be a smooth compact manifold,* V *and* W *smooth vector bundles on* X, U *an open set in* $C^\infty(X, V)$, *and* $P: U \to C^\infty(X, W)$, *a smooth nonlinear elliptic partial differential operator. Let* A *and* B *be LCFC submanifolds of* U *and of* $C^\infty(X, W)$ *respectively, such that the restriction* P_A *of* P *to* A, *sends* A *into* B. *Then the Jacobian criterion holds for* P_A, *namely, if the derivative of* $P_A: A \to B$ *is invertible at* $\varphi_0 \in A$, *then* P_A *is a local diffeomorphism near* φ_0.

This is a convenient variant of the Nash-Moser theorem (e.g. [14]) regarding suitable restrictions of elliptic operators. It is established in a separate paper [11] (see also [22]). It relies only on the *classical* (Banach) inverse function theorem combined with *elliptic regularity.*

Remark 3.2. The Nash-Moser theorem has been studied by many authors, see the bibliography below and further references in [14] [15] [25].

4. PROPERNESS

In view of (2), theorem 3.1 implies that P_λ is open. We want to apply lemma 2.1 in order to prove that P_λ is surjective from A_λ to B_λ. Since $P_\lambda(A_\lambda) \neq \emptyset$ (it contains 0), and since B_λ is connected, this amounts to proving that P_λ is *proper.* Let us explain why *a priori* estimates imply properness.

Concerning subsets in A_λ we have