Zeitschrift: L'Enseignement Mathématique

Band: 34 (1988)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: QUILLEN'S THEOREM ON BUILDINGS AND THE LOOPS ON A SYMMETRIC SPACE

Kapitel: §7. Bott Periodicity

Autor: Mitchell, Stephen A.

DOI: https://doi.org/10.5169/seals-56592

Following the procedure discussed in § 5, we have at once that G_β is all of $SU(n)$, so $K_\beta = K \cong U(n-1)$. Note $K/C_K\mathfrak{t}_{\mathfrak{m}} = S^{2n-3}$. On the other hand $K_{2\beta} \cong SO(2)$ ($G_{2\beta}$ is the $SU(2)$ in the upper left corner). From the Dynkin diagram we conclude that our model for $\Omega \mathbf{C}P^{n-1}$ has one cell in each of the dimensions 0, 1, $2n-2$, $2n-1$, $4n-4$, $4n-3$, ... in other words, the cell series is $(1+t)(1+t^{2n-2})^{-1}$. (Recall that the affine Weyl group of type $\tilde{A}_1$ is just the free product $\mathbf{Z}/2 * \mathbf{Z}/2$, so that the Bruhat cells are indexed by $1, s_0, s_1s_0, s_0s_1s_0$, etc. By the above remarks, s_0 receives weight one and s_1 weight $2n-3$, hence our formula.)

§ 7. Bott Periodicity

Bott's theorem, in its original form [6], is a general statement about the range in which certain maps $K/L \xrightarrow{\varphi} \Omega G/K$ are homotopy equivalences. The periodicity theorems proper are then deduced from this, taking G, K, L to be suitable classical groups. In this section we derive a version of Bott's theorem by showing that in many cases the map φ is a homeomorphism onto a Schubert subspace of $\Omega(G/K)$; then one merely counts cells. In fact, in these cases we will be able to read off the desired range directly from the Dynkin diagram of G/K.

We assume that G is simple and simply-connected. (As usual, the essential point is that G/K is simply-connected; then we can if necessary replace G by its universal cover.) Let $\lambda: [0, 1] \to G$ be a path of the form $\lambda(t) = \exp tX$, where X belongs to the coweight lattice J_m. In otherwords, $X \in \mathfrak{t}_{\mathfrak{m}}$ and $\exp X$ is central in G. Then for all $k \in K$, the path $\varphi_\lambda \equiv \lambda k \lambda^{-1} k^{-1}$ actually lies in $(\Omega_{alg} G)^\tau$; see the proof of 4.2. Hence $\lambda \mapsto \varphi_\lambda$ defines a *Bott map* $K/C_k\lambda \xrightarrow{\varphi} (\Omega_{alg}G)^\tau$ ($\cong \Omega G/K$). Identifying J_m with the group of paths λ as above, the most interesting λ are obviously the fundamental coweights ε_i dual to the simple restricted roots β_i: $\beta_j(\varepsilon_i) = \delta_{ij} (1 \leqslant i, j \leqslant l)$. Among these one may single out the very convenient class of *miniscule coweights.* These are the ε_i dual to a *miniscule root* β_i-i.e. a simple root which occurs with coefficient one in the highest root β_0. The miniscule coweights are precisely the nonzero elements of the coweight lattice which are also vertices of the Cartan simplex. They exist whenever the root system is reduced and not of type G_2, F_4 or E_8; in terms of the Dynkin diagram, they correspond to nodes on the ordinary diagram which are conjugate to the special node $-\alpha_0$ under an automorphism of the extended diagram. Thus for example in type A_n every simple root is miniscule, whereas the number of miniscule roots in types B_n, C_n, D_n, E_6, E_7 is respectively 1, 1, 3, 3, 1. Next, define the *distance* $d(s_i, s_j)$ between two elements of $\tilde{S}_{\mathbf{R}}$ (or nodes on the extended Dynkin diagram of G/K) as follows. Given a path p from s_i to s_j on the extended Dynkin diagram, let m_p be the sum of the multiplicities of the vertices of the path (including s_i and s_j). Then $d(s_i, s_j)$ is the minimal possible value of m_p (p ranging over all paths). For example, in the split case, with $m_\beta = 1$ for all simple restricted roots β, $d(s_i, s_j)$ is just the minimal number of vertices in a path linking s_i to s_j. (Arrows are ignored, and doubled or tripled edges in the diagram are counted as single edges.) We may now state our version of Bott's theorem:

(7.1) THEOREM. *Let ε_i be a miniscule coweight of the restricted root system Σ, and let $\varphi: K/C_K\varepsilon_i \to \Omega G/K$ be the Bott map associated to ε_i^{-1}. Then φ is an isomorphism on homotopy groups in dimensions less than $d(s_0, s_i) - 1$, and is an epimorphism in dimension $d(s_0, s_i) - 1$.*

(7.2) COROLLARY (Bott Periodicity). *There exist Bott maps of the following form, which are isomorphisms on homotopy through the indicated range of dimensions:*

(*a*)	$G^{\mathbf{C}}_{2n,2} \to \Omega SU(2n)$	$(2n)$
(*b*)	$SO(4n)/U(2n) \to \Omega_0 SO(4n)$	$(4n-4)$
(*c*)	$U(2n)/Sp(n) \to \Omega SO(4n)/U(2n)$	$(4n-4)$
(*d*)	$G^{\mathbf{H}}_{2n,n} \to \Omega SU(4n)/Sp(2n)$	$(4n+2)$
(*e*)	$Sp(n)/U(n) \to \Omega Sp(n)$	$(2n)$
(*f*)	$U(n)/O(n) \to \Omega Sp(n)/SU(n)$	(n)
(*g*)	$G^{\mathbf{R}}_{2n,n} \to \Omega SU(2n)/SO(2n)$	$(n-1)$

Proof of Corollary. We need only exhibit miniscule coweights ε_i such that $d(s_0, s_i) - 2$ is the number indicated and $K/C_K\varepsilon_i$ is as shown. We will do this for (*c*) and (*d*) and leave the rest of the fun to the reader (see § 6). In case (*d*), we have seen that Σ has type A_{4n-1} and hence every simple root is miniscule; we also know the multiplicities all equal four. Taking $\varepsilon_i = \varepsilon_n$, we obviously have $d(s_0, s_n) = 4n + 4$. In case (*c*), Σ has type C_{2n}; there is one miniscule root α_{2n}. From (6.2) we compute $d(s_0, s_{2n}) = 4n - 2$. □

Proof of (7.1). The proof is an easy generalization of that of Propositions 2.2 and 2.6 in [25] (note, however, that $d(s_i, s_j)$ is defined somewhat differently there). Therefore it will only be sketched. First of all, consider the set of restricted roots β such that $\beta(\varepsilon) = 0$. This set is spanned by the set I of simple roots it contains, and if I' is the corresponding set in S (as usual), $C_G\varepsilon =$. Thus $C_K\varepsilon = (C_G\varepsilon)^\sigma = K_I$. Since K_I is a maximal compact subgroup of the parabolic $\mathcal{O}_I((=(P_{I'})^\sigma)$, the Iwasawa decomposition $\mathcal{O}_I = K_I Q$ shows that $K/C_K\varepsilon = G_{\mathbf{R}}/\mathcal{O}_I$. Since $(\Omega_{alg})^\tau = \tilde{G}_{\mathbf{R}}/P^\tau$, the Bott map can be thought of as a map $G_{\mathbf{R}}/\mathcal{O}_I \to \tilde{G}_{\mathbf{R}}/P^\tau$. To describe this map in terms of Bruhat cells we need to alter it slightly. First, let $y_i = \varepsilon_i w$, where $w = w_{[i]} w_0 \in W_{\mathbf{R}}$. Here $W_{[i]}$ denotes the maximal length element of $W_{[i]}$, where $[i] = S_{\mathbf{R}} - \{i\}$. (This definition is due to Iwahori and Matsumoto [16], among other things it provides a splitting of the projection $\tilde{W} \to \tilde{W}/\tilde{W}$.) Then the map $\varphi' : K/C_K\varepsilon \to (L_{alg}G)^\tau/K = (\Omega_{alg}G)^\tau$ given by $k \mapsto \mu_i^{-1} k \mu_i$ is homotopic to φ, since $\varphi' = w^{-1}\varphi$ and K is connected. Hence in the proof we may replace φ by φ'. The point of this is:

(7.3) LEMMA.

(a) *The map* $\Theta : f \mapsto \mu_i^{-1} f \mu_i$ *defines an automorphism of* $\tilde{G}_{\mathbf{C}}$ *preserving* $\tilde{G}_{\mathbf{R}}$.

(b) $\Theta : \tilde{G}_{\mathbf{R}} \to \tilde{G}_{\mathbf{R}}$ *preserves* $\tilde{Q}$, *and in fact permutes the simple roots (defining an automorphism of the extended Dynkin diagram). In particular* $\mu_i \cdot (1, \beta_0) = (0, -\beta_i)$.

(c) $\Theta \mid_{G_{\mathbf{R}}}$ *induces an embedding* $G_{\mathbf{R}}/\mathcal{O}_I \to \tilde{G}_{\mathbf{R}}/P^\tau$, *which corresponds to* φ' *and is a homeomorphism onto a Schubert subspace.*

Remarks. In (a) we have identified $\tilde{G}_{\mathbf{C}}$ with the group of paths: $[0, 1] \to G_{\mathbf{C}}$ of the form $f(e^{2\pi i t})$, where $f : S^1 \to G_{\mathbf{C}}$ is algebraic. In (b), the automorphism of the Dynkin diagram preserves multiplicities.

It remains to show that every cell not in the image of φ' has dimension at least $d(s_0, s_j)$. Now Θ preserves the simple reflections $\tilde{S}_{G/K}$, with $\Theta(s_i) = s_0$, and clearly the cells which are in the image of φ' are precisely the E_w such that $w \in W^{S_{\mathbf{R}}}$ and $\Theta(s_0)$ does not occur in a reduced expression for w. Since every such expression must begin on the right with s_0, a moments reflection should convince the reader that the minimal dimension of a cell involving $\Theta(s_0)$ is $d(s_0, \Theta(s_0))$. Since

$$d(s_0, \Theta(s_0)) = d(\Theta^{-1}(s_0), s_0) = d(s_0, s_i),$$

this completes the proof. □