L'Enseignement Mathématique, t. 34 (1988), p. 167-204

ISOCLINIC n-PLANES IN R^{2n} AND THE HOPF-STEENROD SPHERE BUNDLES $S^{2n-1} \to S^n$, $n = 2, 4, 8$

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0. INTRODUCTION

The construction of the sphere bundles $S^{2n-1} \to S^n$, $n = 2, 4, 8$, by N. Steenrod was accomplished in an ingenious but rather roundabout way, using the famous Hopf maps and the systems of complex numbers, quaternions and Cayley numbers (cf. Hopf [2], Steenrod [5, pp. 105-110] and Hilton [1, pp. 51-55]). In this paper, we show how the theory of mutually isoclinic n-planes in a real Euclidean $2n$-space R^{2n} as developed by Wong in [8, 9] enables us to reconstruct these sphere bundles in a more natural manner by working strictly within the field of real numbers and giving the three cases $n = 2, 4, 8$ a more unified treatment. In addition, we prove that the bundle group $O(8)$ of the Hopf-Steenrod sphere bundle $S^{15} \to S^8$ can be replaced by $SO(8)$ but not by any subgroup of $SO(8)$.

In § 1, we recall certain results on maximal sets of mutually isoclinic n-planes in R^{2n} that motivated our investigation. In § 2, we confine ourselves to the cases $n = 2, 4, 8$, and prove some results that will be used later. In § 3, we construct three sphere bundles by using maximal sets of mutually isoclinic n-planes in R^{2n}. In § 4, we give a unified and explicit formulation of the three Hopf-Steenrod sphere bundles, using as Steenrod did the Hopf maps and systems of complex numbers, quaternions and Cayley numbers. In § 5, we prove that the Hopf maps and maximal sets of mutually isoclinic n-planes in R^{2n}, $n = 2, 4, 8$, are equivalent concepts, and that the reformulated Hopf-Steenrod sphere bundles described in § 4 are topologically essentially the same as the sphere bundles constructed in § 3. The paper ends with two appendices in which we explain the operations of Cayley numbers, and give a direct proof that for $n = 2, 4$, or 8, the n-planes in R^{2n} containing the Hopf fibers of S^{2n-1} are mutually isoclinic n-planes.

In a continuation of this paper being prepared, we shall show that the image of the Hopf fibers of S^{2n-1}, $n = 2, 4$, or 8, under an inversion in R^{2n} has some very interesting properties which include those recently found by J. B. Wilker [7] for the case $n = 2$.

We wish to thank Prof. Wilker for letting us have a preprint of his paper, and Prof. Kee-Yuen Lam for some helpful discussions.

1. Some results on isoclinic n-planes in R^{2n}

By a Euclidean (vector) m-space R^m, where m is a positive integer, we mean an m-dimensional vector space provided with a positive definite inner product. An r-plane $(1 \leqslant r \leqslant m-1)$ in R^m is an r-dimensional vector subspace of R^m provided with the induced inner product. In R^m, length of a vector, angle between two vectors, orthogonality between a k-plane and an r-plane, (orthogonal) projection of a vector on an r-plane, orthonormal bases and rectangular coordinates are defined in the usual way.

In an R^{2n}, let $\mathbf{A}$, $\mathbf{B}$ be any two n-planes. Then we say that $\mathbf{A}$ is *isoclinic* with $\mathbf{B}$ at angle θ if the angle between every nonzero vector in $\mathbf{A}$ and its projection on $\mathbf{B}$ is always equal to θ. It turns out that if $\mathbf{A}$ is isoclinic with $\mathbf{B}$ at angle θ, then $\mathbf{B}$ is isoclinic with $\mathbf{A}$ at the same angle θ. Therefore, in this case, we shall say that $\mathbf{A}$ and $\mathbf{B}$ are isoclinic at angle θ, or simply, $\mathbf{A}$ and $\mathbf{B}$ are isoclinic.

A set Φ of n-planes in R^{2n} is said to be a *maximal set of mutually isoclinic n-planes* if every pair of n-planes in Φ are isoclinic and Φ is not contained in a larger set of mutually isoclinic n-planes. It is easy to see from definition that if $\mathbf{A}$ is isoclinic with $\mathbf{B}$ at angle θ, then its orthogonal complement $\mathbf{A}^{\perp}$ is isoclinic with $\mathbf{B}$ at angle $\frac{\pi}{2} - \theta$. Consequently, if Φ is any maximal set of mutually isoclinic n-planes in R^{2n} and $\mathbf{A} \in \Phi$, then $\mathbf{A}^{\perp} \in \Phi$.

In his memoir [8] Wong determined, for each n, the dimensions of the maximal sets of mutually isoclinic n-planes in R^{2n}, the number of non-congruent maximal sets of a given dimension, and explicit equations of the n-planes in any maximal set of mutually isoclinic n-planes containing a given n-plane.

In the following, we summarize some of his results related to the problem studied in this paper.

THEOREM 1.1. (Wong [8, pp. 25-26]). *In R^{2n} provided with a rectangular coordinate system $(x, y) \equiv ([x_1 \dots x_n], [x_{n+1} \dots x_{2n}])$, any maximal set Φ of mutually isoclinic n-planes containing the n-plane $\mathbf{O}: y = 0$ (and consequently, also the n-plane $\mathbf{O}^{\perp}: x=0$) is congruent to the set of n-planes with equations*

$$(1.1) \qquad x = 0, \quad \text{or} \quad y = x(\lambda_0 + \lambda_1 B_1 + \dots + \lambda_{p-1} B_{p-1}),$$

where $(\lambda_0, \lambda_1, \dots, \lambda_{p-1})$ *are* p *real parameters and* $(B_1, \dots, B_{p-1})$ *is a maximal set of* $n \times n$ *matrices satisfying the Hurwitz matrix equations*

$$(1.2) \quad B_h + B_h^T = 0, \quad B_h^2 = -I, \quad B_h B_k + B_k B_h = 0\,(h, k = 1, \dots; h \neq k).$$

Here, by $(B_1, \dots, B_{p-1})$ being a maximal set of matrices satisfying equations (1.2), we mean that $(B_1, \dots, B_{p-1})$ is not a subset of another set containing more matrices satisfying equations (1.2).

Remark. It is of some historical interest that equations (1.2) first appeared in the literature in 1923 in connection with the famous problem of A. Hurwitz [4] on composition of quadratic forms, and then reappeared in 1961 in a very different type of problem. For more information about these equations, we refer the reader to Wong's memoir [8] and J. A. Tyrrell-J. G. Semple's book [6].

A maximal set of mutually isoclinic n-planes in R^{2n} is said to be p-dimensional (or, of dimension p), if it contains p parameters $\lambda_0, \lambda_1, \dots, \lambda_{p-1}$ as in Theorem 1.1. It can be proved (cf. [8, p. 54]) that the dimension of a maximal set of mutually isoclinic n-planes in R^{2n} is always $\leqslant n$, and that there exist maximal sets of dimension n in R^{2n} if and only if $n = 2, 4$, or 8. Moreover, we have

Theorem 1.2. (Wong [8, p. 57]). *Let* Φ *be a p-dimensional maximal set of mutually isoclinic n-planes in* R^{2n}. *Then, through any point in* $R^{2n}\backslash O$, *there passes at most one n-plane of* Φ. *In order that through any point in* $R^{2n}\backslash O$, *there passes exactly one n-plane of* Φ, *it is necessary and sufficient that* $n = p = 2, 4,$ *or* 8.

Theorem 1.3. (Wong [8, pp. 62-64]). *Any p-dimensional maximal set of mutually isoclinic n-planes in* R^{2n}, *if regarded as a submanifold of the Grassmann manifold of n-planes in* R^{2n}, *is diffeomorphic with the p-sphere* S^p.

Since the unit sphere S^{2n-1} in R^{2n} is intersected by an n-plane in a great $(n-1)$-sphere, a consequence of Theorems 1.2 and 1.3 is

Theorem 1.4. (Wong [8, pp. 65-66]). *In* R^{2n}, $n = 2, 4,$ *or* 8, *the intersection of the unit sphere* S^{2n-1} *by any n-dimensional maximal set of mutually isoclinic n-planes furnishes a fibering of* S^{2n-1} *by* S^{n-1} *over* S^n.

The above three theorems direct our attention to the three special cases $n = 2, 4, 8$, for which we now prove:

THEOREM 1.5.

(i) *For* $n = 2$, *every maximal real solution of the Hurwitz matrix equations (1.2) is orthogonally similar to the maximal solution* $\{B_1\}$ *where*

$$B_1 = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}. \tag{1.3}$$

(ii) *For* $n = 4$, *every maximal real solution of the Hurwitz matrix equations (1.2) is orthogonally similar to the maximal solution* $\{B_1, B_2, B_3\}$ *where*

$$B_1 = \begin{bmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix}, \quad B_2 = \begin{bmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{bmatrix}, \quad B_3 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}. \tag{1.4}$$

(iii) *For* $n = 8$, *every maximal real solution of the Hurwitz matrix equations (1.2) contains either* 3 *or* 7 *matrices. In the latter case, it is orthogonally similar to the maximal solution* $\{B_1, ..., B_7\}$, *where*

$$B_1 = \begin{bmatrix} & 1 & & & & & & \\ -1 & & & & & & & \\ & & & 1 & & & & \\ & & -1 & & & & & \\ & & & & & 1 & & \\ & & & & -1 & & & \\ & & & & & & & -1 \\ & & & & & & 1 & \end{bmatrix}, \quad B_2 = \begin{bmatrix} & & 1 & & & & & \\ & & & -1 & & & & \\ -1 & & & & & & & \\ & 1 & & & & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ & & & & -1 & & & \\ & & & & & -1 & & \end{bmatrix}, \tag{1.5}$$

$$B_3 = \begin{bmatrix} & & & 1 & & & & \\ & & 1 & & & & & \\ & -1 & & & & & & \\ -1 & & & & & & & \\ & & & & & & & 1 \\ & & & & & & -1 & \\ & & & & & 1 & & \\ & & & & -1 & & & \end{bmatrix}, \quad B_4 = \begin{bmatrix} & & & & 1 & & & \\ & & & & & -1 & & \\ & & & & & & -1 & \\ & & & & & & & -1 \\ -1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \end{bmatrix},$$

$$B_5 = \begin{bmatrix} & & & & & 1 & & \\ & & & & 1 & & & \\ & & & & & & & -1 \\ & & & & & & 1 & \\ & -1 & & & & & & \\ -1 & & & & & & & \\ & & & -1 & & & & \\ & & 1 & & & & & \end{bmatrix}, \quad B_6 = \begin{bmatrix} & & & & & & 1 & \\ & & & & & & & 1 \\ & & & & 1 & & & \\ & & & & & -1 & & \\ & & -1 & & & & & \\ & & & 1 & & & & \\ -1 & & & & & & & \\ & -1 & & & & & & \end{bmatrix},$$

$$B_7 = \begin{bmatrix} & & & & & & & 1 \\ & & & & & & -1 & \\ & & & & & 1 & & \\ & & & & 1 & & & \\ & & & -1 & & & & \\ & & -1 & & & & & \\ & 1 & & & & & & \\ -1 & & & & & & & \end{bmatrix}.$$

Proof. This theorem is a reformulation of Theorem 8.1 in [8, pp. 107-109]. In fact, if we denote by C_i the matrices used in Theorem 7.2 in [8, pp. 54-56] and by U the diagonal matrix $(1, -1, \ldots, -1)$ of order n, then we can easily verify that $B_i = UC_iU^{-1}$.

An immediate consequence of Theorem 1.1 and 1.5 is the following

THEOREM 1.6.

(i) *In* R^4, *every maximal set of mutually isoclinic* 2-*planes is of dimension* 2 *and is congruent to the set* Φ_2 *consisting of the* 2-*plane* $x = 0$ *and the* 2-*planes* $y = xB(\lambda)$, *where*

$$B(\lambda) \equiv \lambda_0 + \lambda_1 B_1 = \begin{bmatrix} \lambda_0 & \lambda_1 \\ -\lambda_1 & \lambda_0 \end{bmatrix}. \tag{1.6}$$

(ii) *In* R^8, *every maximal set of mutually isoclinic* 4-*planes is of dimension* 4 *and is congruent to the set* Φ_4 *consisting of the* 4-*plane* $x = 0$ *and the* 4-*planes* $y = xB(\lambda)$, *where*

$$B(\lambda) \equiv \lambda_0 + \lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 B_3 = \begin{bmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\ -\lambda_1 & \lambda_0 & \lambda_3 & -\lambda_2 \\ -\lambda_2 & -\lambda_3 & \lambda_0 & \lambda_1 \\ -\lambda_3 & \lambda_2 & -\lambda_1 & \lambda_0 \end{bmatrix}, \tag{1.7}$$

(iii) *In* R^{16}, *every maximal set of mutually isoclinic* 8-*planes is of dimension* 4 *or* 8. *Every maximal set of dimension* 8 *is congruent to the set* Φ_8 *consisting of the* 8-*plane* $x = 0$ *and the* 8-*planes* $y = xB(\lambda)$, *where*

$$(1.8)\qquad B(\lambda) \equiv \lambda_0 + \lambda_1 B_1 + \dots + \lambda_7 B_7 = \begin{bmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 \\ -\lambda_1 & \lambda_0 & \lambda_3 & -\lambda_2 & \lambda_5 & -\lambda_4 & -\lambda_7 & \lambda_6 \\ -\lambda_2 & -\lambda_3 & \lambda_0 & \lambda_1 & \lambda_6 & \lambda_7 & -\lambda_4 & -\lambda_5 \\ -\lambda_3 & \lambda_2 & -\lambda_1 & \lambda_0 & \lambda_7 & -\lambda_6 & \lambda_5 & -\lambda_4 \\ -\lambda_4 & -\lambda_5 & -\lambda_6 & -\lambda_7 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\ -\lambda_5 & \lambda_4 & -\lambda_7 & \lambda_6 & -\lambda_1 & \lambda_0 & -\lambda_3 & \lambda_2 \\ -\lambda_6 & \lambda_7 & \lambda_4 & -\lambda_5 & -\lambda_2 & \lambda_3 & \lambda_0 & -\lambda_1 \\ -\lambda_7 & -\lambda_6 & \lambda_5 & \lambda_4 & -\lambda_3 & -\lambda_2 & \lambda_1 & \lambda_0 \end{bmatrix}$$

In (1.6), (1.7) and (1.8) above, the λ_0 in $\lambda_0 + \lambda_1 B_1 + \dots$ stands for the scalar matrix $\lambda_0 I$.

REMARK. The maximal set Φ_n of mutually isoclinic n-planes in R^{2n} in Theorem 1.6 is congruent to that in Theorem 7.2 in [8, pp. 54-56] under the orthogonal transformation

$$f:(x_1, x_2, \dots, x_n, x_{n+1}, x_{n+2}, \dots, x_{2n})$$
$$\to (x_1, -x_2, \dots, -x_n, x_{n+1}, -x_{n+2}, \dots, -x_{2n}),$$

which obviously leaves invariant the n-planes $\mathbf{O}: y = 0$ and $\mathbf{O}^\perp : x = 0$. To see this, let us denote by Ψ_n the maximal set of mutually isoclinic n-planes in Theorem 7.2 in [8, pp. 54-56] and write the equations of these n-planes as $x = 0$ and

$$y = x(\lambda_0 + \lambda_1 C_1 + \dots + \lambda_{n-1} C_{n-1}).$$

Then f sends Ψ_n to the set $f(\Psi_n)$ of mutually isoclinic n-planes with equations $x = 0$ and

$$yU = xU(\lambda_0 + \lambda_1 C_1 + \dots + \lambda_{n-1} C_{n-1}),$$

i.e.,

$$y = xU(\lambda_0 + \lambda_1 C_1 + \dots + \lambda_{n-1} C_{n-1})U^{-1},$$

where U is the diagonal matrix $(1, -1, \dots, -1)$ of order n. But, as we have seen in the proof of Theorem 1.5, these equations are the same as $x = 0$ and

$$y = x(\lambda_0 + \lambda_1 B_1 + \dots + \lambda_{n-1} B_{n-1}).$$

Therefore, $f(\Psi_n)$ is the set Φ_n of mutually isoclinic n-planes in our Theorem 1.6.

2. Some further results

From now on we shall confine our attention to n-dimensional maximal sets of mutually isoclinic n-planes in R^{2n}, and therefore, n has always the values 2, 4, or 8 unless stated otherwise.

In this section, we prove a few more theorems for use in § 3. In these theorems, the indices a, b have the range of values $(0, 1, ..., n-1)$; $B_0 = I$ is the identity matrix of order n; $B_1, ..., B_{n-1}$ are the $n \times n$ matrices listed in Theorems 1.5 and 1.6; $\lambda = (\lambda_a)$ is an ordered set of n real parameters; and

$$B(\lambda) \equiv \sum_a \lambda_a B_a, \qquad N(\lambda) \equiv \sum_a \lambda_a^2 .$$

Moreover, for any matrix M, we denote its transpose by M^T.

Theorem 2.1.

(i) $B(\lambda)B(\lambda)^T = N(\lambda)I.$

(ii) *If* $\lambda \neq 0$, *then*

$$B(\lambda)^{-1} = B(\lambda)^T/N(\lambda) = \sum_a \lambda_a B_a^T/N(\lambda),$$

so that if $\lambda \neq 0$, *the equation* $y = xB(\lambda)$ *is equivalent to the equation* $x = yB(\mu)^T$, *where* $\mu = \lambda/N(\lambda) \neq 0$.

(iii) $$\det B(\lambda) = +\big(N(\lambda)\big)^{n/2} .$$

(iv) *If* $N(\lambda) = 1$, *then* $B(\lambda) \in SO(n)$, *where* $SO(n)$ *is the set of all orthogonal matrices of order* n *and determinant* $+1$.

Proof.
$$\begin{aligned} B(\lambda)B(\lambda)^T &= \big(\textstyle\sum_a \lambda_a B_a\big)\big(\sum_b \lambda_b B_b^T\big) = \sum_{a,b} \lambda_a \lambda_b B_a B_b^T \\ &= \textstyle\sum_a \lambda_a^2 B_a B_a^T + \sum_{a<b} \lambda_a \lambda_b (B_a B_b^T + B_b B_a^T), \end{aligned}$$

which, on account of the Hurwitz matrix equations (1.2), is equal to $(\sum_a \lambda_a^2)I = N(\lambda)I$. This proves (i), and also (ii). To prove (iii), we first note that since $B(\lambda)$ is a square matrix of order n, $\det B(\lambda)$ is a homogeneous polynomial of degree n in the λ_a's, and it follows from (i) that

$$(\det B(\lambda))^2 = \det\big(B(\lambda)B(\lambda)^T\big) = \big(N(\lambda)\big)^n .$$

Therefore,

$$\text{(2.1)} \qquad \det B(\lambda) = \pm (N(\lambda))^{n/2} = \pm (\lambda_0^2+\lambda_1^2+\ldots+\lambda_{n-1}^2)^{n/2}$$
$$= \pm (\lambda_0^n + \text{other product terms in } \lambda_a)\,.$$

On the other hand, since $B_0 = I$, and $B_1, \ldots, B_{n-1}$ are all skew-symmetric matrices, the diagonal elements of $B(\lambda)$ are all equal to λ_0, and none of the other elements of $B(\lambda)$ is equal to λ_0. Therefore,

$$\det B(\lambda) = \lambda_0^n + \text{other product terms in } \lambda_a\,.$$

Comparison of this with (2.1) gives (iii). Finally, (iv) follows immediately from (i) and (iii).

Returning to Theorems 1.2 and 1.6, we now prove

THEOREM 2.2. *Let* Φ_n *be the maximal set of mutually isoclinic n-planes in* R^{2n} *described in Theorem 1.6, and let* (u, v) *be any vector in* R^{2n}. *If* $u \neq 0$, *then the unique n-plane in* Φ_n *containing* (u, v) *is*

$$\text{(2.2)} \qquad y = x[vu^T-(vB_1u^T)B_1-\ldots-(vB_{n-1}u^T)B_{n-1}]/(uu)^T\,.$$

If $v \neq 0$, *then the unique n-plane in* Φ_n *containing* (u, v) *is*

$$\text{(2.3)} \qquad x = y[uv^T-(uB_1^Tv^T)B_1^T-\ldots-(uB_{n-1}^Tv^T)B_{n-1}^T]/(vv)^T\,.$$

Here, $B_1, \ldots, B_{n-1}$ *are the matrices in (1.3), (1.4), or (1.5) according as* $n = 2, 4,$ *or* 8.

Proof. We shall prove only (2.2) for the case $u \neq 0$, as (2.3) for the case $v \neq 0$ can be proved similarly. Suppose that $u \neq 0$ and

$$\text{(2.4)} \qquad y = x(\lambda_0+\lambda_1B_1+\ldots+\lambda_{n-1}B_{n-1})$$

is an n-plane in Φ_n containing (u, v). Then we have

$$v = u(\lambda_0+\lambda_1B_1+\ldots+\lambda_{n-1}B_{n-1})\,,$$

which can be written as

$$v = [\lambda_0\lambda_1 \ldots \lambda_{n-1}] \begin{bmatrix} u \\ \\ uB_1 \\ \vdots \\ uB_{n-1} \end{bmatrix} .$$

Multiplying the two sides of this equation on the right by

$$[u^T, -B_1u^T, ..., -B_{n-1}u^T]$$

and making use of the Hurwitz matrix equations (1.2), we get

$$v[u^T, -B_1u^T, ..., -B_{n-1}u^T] = [\lambda_0\lambda_1 \dots \lambda_{n-1}]\,(uu^T)I\,.$$

Since $uu^T \neq 0$, the above equation determines the λ_a's uniquely in terms of u, v. Now with these values of λ_a's, equation (2.4) becomes equation (2.2), as we wanted to prove. Incidentally, the above proof also confirms that there is exactly one n-plane in Φ_n containing the vector (u, v) (cf. Theorem 1.2).

Next, we give a direct proof of Theorem 1.3 for the special cases $n = 2, 4, 8$, and state the result as

THEOREM 2.3. *The maximal set* $\Phi_n = \{x = 0, y = xB(\lambda)\}$ *of mutually isoclinic n-planes in* R^{2n}, $n = 2, 4,$ *or* 8, *can be given a differentiable structure so that it is diffeomorphic with the n-sphere* S^n.

Proof. Let us regard Φ_n as a point set whose elements are the n-planes in Φ_n. Then, the subset $\Phi_n \backslash \mathbf{O}^\perp = \{y = xB(\lambda)\}$ of Φ_n is an open subset in which we can define a coordinate system by assigning to the element $y = xB(\lambda)$ the coordinate $\lambda = (\lambda_0, \lambda_1, ..., \lambda_{n-1})$. The subset $\Phi_n \backslash \mathbf{O} = \{x = 0$ and $y = xB(\lambda)$, where $\lambda \neq 0\}$ of Φ_n is also an open subset. By Theorem 2.1 (ii), this subset is the same as the subset $\{x = yB(\mu)^T\}$, and so, we can define in it a coordinate system by assigning to the element $x = yB(\mu)^T$ the coordinate $\mu = (\mu_0, \mu_1, ..., \mu_{n-1})$. Thus Φ_n is covered by the two coordinate neighborhoods

$$(\Phi_n \backslash \mathbf{O}^\perp, \lambda)\,, \quad (\Phi_n \backslash \mathbf{O}, \mu)\,. \tag{2.5}$$

Moreover, we can see from Theorem 2.1 (ii) that for any element in $(\Phi_n \backslash \mathbf{O}^\perp) \cap (\Phi_n \backslash \mathbf{O}) = \Phi_n \backslash \{\mathbf{O}^\perp, \mathbf{O}\}$, its two coordinates λ, μ, both nonzero, are related by

$$\mu = \lambda/N(\lambda), \text{ or equivalently, } \lambda = \mu/N(\mu)\,. \tag{2.6}$$

Hence, Φ_n is an n-dimensional manifold.

To show that Φ_n is diffeomorphic with the n-sphere S^n, we view S^n as the unit sphere $x_1^2 + \dots + x_{n+1}^2 = 1$ in R^{n+1}, and use stereographic projections. Let $q_1(0, ..., 0, 1)$ and $q_2(0, ..., 0, -1)$ be respectively the north and south poles of S^n. Then S^n is the union of the two open subsets

$S^n \backslash q_1$ and $S^n \backslash q_2$. For an arbitrary point q in $S^n \backslash q_1$, let the line $q_1 q$ meet the equator n-plane $x_{n+1} = 0$ at the point $(\lambda, 0)$; and for an arbitrary point q in $S^n \backslash q_2$, let the line $q_2 q$ meet the equator n-plane $x_{n+1} = 0$ at the point $(\mu, 0)$. Then S^n is covered by the two coordinate neighborhoods

$$(S^n \backslash q_1, \lambda), \quad (S^n \backslash q_2, \mu). \tag{2.7}$$

Moreover, it is easy to verify that for a point in $S^n \backslash \{q_1, q_2\}$, its two coordinates λ and μ are also both nonzero and related by (2.6).

It now follows from (2.5), (2.6) and (2.7) that if f_1 is the map from $\Phi_n \backslash \mathbf{O}^\perp$ to $S^n \backslash q_1$ sending an n-plane in $\Phi_n \backslash \mathbf{O}^\perp$ with coordinate λ to the point in $S^n \backslash q_1$ with the same coordinate λ, and f_2 is the map from $\Phi_n \backslash \mathbf{O}$ to $S^n \backslash q_2$ sending an n-plane in $\Phi_n \backslash \mathbf{O}$ with coordinate μ to the point in $S^n \backslash q_2$ with the same coordinate μ, then f_1, f_2 combined will give a diffeomorphism from Φ_n to S^n.

In the remainder of this section, we are concerned exclusively with the matrices $B(\lambda)$ with $N(\lambda) = 1$. For convenience, we shall denote such matrices by $B(\lambda')$, with the understanding that λ' always satisfies the condition $N(\lambda') = 1$.

We know from Theorem 2.1 (iv) that every $B(\lambda')$ belongs to $SO(n)$. Let us now regard $SO(n)$ as the special orthogonal group. Then the set of elements $B(\lambda')$ of $SO(n)$ will generate a subgroup of $SO(n)$. We wish to know what this subgroup of $SO(n)$ is, and the next three theorems will give us the answer.

THEOREM 2.4. *For* $n = 2$, *the set of elements* $B(\lambda')$ *forms the group* $SO(2)$ *which is isomorphic with* S^1.

Proof. Since

$$B(\lambda') = \begin{bmatrix} \lambda'_0 & \lambda'_1 \\ -\lambda'_1 & \lambda'_0 \end{bmatrix} \quad \text{and} \quad \det B(\lambda') = (\lambda'_0)^2 + (\lambda'_1)^2 = 1 ,$$

the elements of $SO(2)$ are the elements $B(\lambda')$ themselves.

THEOREM 2.5. *For* $n = 4$, *the set of elements* $B(\lambda')$ *forms a* 3-*parameter subgroup of* $SO(4)$, *isomorphic with* S^3.

Proof. First, since $N(\lambda') = (\lambda'_0)^2 + \ldots + (\lambda'_3)^2 = 1$, the set $B(\lambda')$, with a natural topology, is homeomorphic with the unit 3-sphere S^3 in R^4. Next, using (1.4), we can easily verify that

$$B_2B_3 = -B_1, \quad B_3B_1 = -B_2, \quad B_1B_2 = -B_3.$$

With this and Theorem 2.1 (ii), straight forward computation will show that for any two elements $B(\lambda')$ and $B(\mu')$ of $SO(4)$, the product $B(\lambda')B(\mu')^{-1}$ is an element of $SO(4)$ of the form $B(\nu')$, where the components of ν' are analytic functions of the components of λ' and μ'. This proves our theorem.

For the case $n = 8$, we first observe that the elements $B(\lambda')$ of $SO(8)$ do not, by themselves, form a subgroup of $SO(8)$. For example, although B_1, B_2 are both of the form $B(\lambda')$, their product B_1B_2 is not. In fact, we have

THEOREM 2.6. *For $n = 8$, the set of elements $B(\lambda')$ of $SO(8)$ generates the group $SO(8)$ itself.*

Proof. Our proof consists of two steps (i) and (ii). In (i), we prove that the 28 skew-symmetric 8×8 matrices B_i, $B_iB_j(i, j = 1, \ldots, 7$, and $i<j)$ are linearly independent. In (ii), we prove that the Lie algebra of the subgroup of $SO(8)$ generated by the elements $B(\lambda')$ coincides with the Lie algebra $o(8)$ of $SO(8)$. The assertion in our theorem then follows from the well-known fact in Lie groups that there is a one-one correspondence between the connected Lie subgroups of a Lie group G and the Lie subalgebras of the Lie algebra of G.

(i) From (1.5), we see that the 8×8 matrices $B_i(i = 1, \ldots, 7)$ can be partitioned as

$$B_1 = \begin{bmatrix} J & & & \\ & J & & \\ & & J & \\ & & & -J \end{bmatrix}, \quad B_2 = \begin{bmatrix} & K & & \\ -K & & & \\ & & & I \\ & & -I & \end{bmatrix}, \quad B_3 = \begin{bmatrix} & L & & \\ -L & & & \\ & & & J \\ & & J & \end{bmatrix}$$

$$B_4 = \begin{bmatrix} & & K & \\ & & & -I \\ -K & & & \\ & I & & \end{bmatrix}, \quad B_5 = \begin{bmatrix} & & L & \\ & & & -J \\ -L & & & \\ & -J & & \end{bmatrix},$$

$$B_6 = \begin{bmatrix} & & & I \\ & & K & \\ & -K & & \\ -I & & & \end{bmatrix}, \quad B_7 = \begin{bmatrix} & & & J \\ & & L & \\ & -L & & \\ J & & & \end{bmatrix},$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

are 2×2 submatrices and each empty space represents a 2×2 zero-matrix 0.

Since the matrices I, J, K, L have the properties

$$I^2 = I, \quad J^2 = -I, \quad K^2 = I, \quad L^2 = I,$$
$$JK = -KJ = -L, \quad KL = -LK = J, \quad LJ = -JL = -K,$$

we can easily verify that the products $B_iB_j (i, j = 1, \ldots, 7, \text{ and } i<j)$ are matrices of the same form as B_i, having some of O, $\pm I$, $\pm J$, $\pm K$, $\pm L$ as 2×2 submatrices.

To prove that the 28 matrices B_i, B_iB_j are linearly independent, we construct the 8×8 matrix

$$M \equiv \textstyle\sum_i \alpha_i B_i + \sum\limits_{i<j} \alpha_{ij}(B_iB_j),$$

where the α's are some real numbers, and show that if $M = 0$, then all the α's are zero. Let $M = [M_{hk}]$, where $M_{hk} (h, k = 1, 2, 3, 4)$ are the 2×2 submatrices of M. Then by using the explicit forms of B_i and B_iB_j, we can write M as the sum of the following four matrices:

$$\begin{bmatrix} M_{11} & & & \\ & M_{22} & & \\ & & M_{33} & \\ & & & M_{44} \end{bmatrix} = \alpha_1 \begin{bmatrix} J & & & \\ & J & & \\ & & J & \\ & & & -J \end{bmatrix} + \alpha_{23} \begin{bmatrix} -J & & & \\ & -J & & \\ & & J & \\ & & & -J \end{bmatrix}$$

$$+ \alpha_{45} \begin{bmatrix} -J & & & \\ & J & & \\ & & -J & \\ & & & -J \end{bmatrix} + \alpha_{67} \begin{bmatrix} J & & & \\ & -J & & \\ & & -J & \\ & & & -J \end{bmatrix},$$

$$\begin{bmatrix} & M_{12} & & \\ M_{21} & & & \\ & & & M_{34} \\ & & M_{43} & \end{bmatrix} = \alpha_2 \begin{bmatrix} & K & & \\ -K & & & \\ & & & I \\ & & -I & \end{bmatrix} + \alpha_{13} \begin{bmatrix} & K & & \\ -K & & & \\ & & & -I \\ & & I & \end{bmatrix}$$

$$+ \alpha_3 \begin{bmatrix} & L & & \\ -L & & & \\ & & & J \\ & & J & \end{bmatrix} + \alpha_{12} \begin{bmatrix} & -L & & \\ L & & & \\ & & & J \\ & & J & \end{bmatrix}$$

$$+ \alpha_{46} \begin{bmatrix} & -I & & \\ I & & & \\ & & & -K \\ & & K & \end{bmatrix} + \alpha_{57} \begin{bmatrix} & -I & & \\ I & & & \\ & & & K \\ & & -K & \end{bmatrix}$$

$$+ \alpha_{47} \begin{bmatrix} & -J & & \\ -J & & & \\ & & & -L \\ & & L & \end{bmatrix} + \alpha_{56} \begin{bmatrix} & J & & \\ J & & & \\ & & & -L \\ & & L & \end{bmatrix},$$

$$\begin{bmatrix} & & M_{13} & \\ & & & M_{24} \\ M_{31} & & & \\ & M_{42} & & \end{bmatrix} = \alpha_4 \begin{bmatrix} & & K & \\ & & & -I \\ -K & & & \\ & I & & \end{bmatrix} + \alpha_{15} \begin{bmatrix} & & K & \\ & & & I \\ -K & & & \\ & -I & & \end{bmatrix}$$

$$+ \alpha_5 \begin{bmatrix} & & L & \\ & & & -J \\ -L & & & \\ & -J & & \end{bmatrix} + \alpha_{14} \begin{bmatrix} & & -L & \\ & & & -J \\ L & & & \\ & -J & & \end{bmatrix}$$

$$+ \alpha_{26} \begin{bmatrix} & & I & \\ & & & -K \\ -I & & & \\ & K & & \end{bmatrix} + \alpha_{37} \begin{bmatrix} & & I & \\ & & & K \\ -I & & & \\ & -K & & \end{bmatrix}$$

$$+ \alpha_{27} \begin{bmatrix} & & J & \\ & & & -L \\ J & & & \\ & L & & \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & -J & \\ & & & -L \\ -J & & & \\ & L & & \end{bmatrix}$$

$$\begin{bmatrix} & & & M_{14} \\ & & M_{23} & \\ & M_{32} & & \\ M_{41} & & & \end{bmatrix} = \alpha_6 \begin{bmatrix} & & & I \\ & & K & \\ & -K & & \\ -I & & & \end{bmatrix} + \alpha_{17} \begin{bmatrix} & & & -I \\ & & K & \\ & -K & & \\ I & & & \end{bmatrix}$$

$$+ \alpha_7 \begin{bmatrix} & & & J \\ & & L & \\ & -L & & \\ J & & & \end{bmatrix} + \alpha_{16} \begin{bmatrix} & & & J \\ & & -L & \\ & L & & \\ J & & & \end{bmatrix}$$

$$+ \alpha_{24} \begin{bmatrix} & & & -K \\ & & -I & \\ & I & & \\ K & & & \end{bmatrix} + \alpha_{35} \begin{bmatrix} & & & K \\ & & -I & \\ & I & & \\ -K & & & \end{bmatrix}$$

$$+ \alpha_{25} \begin{bmatrix} & & & -L \\ & & -J & \\ & -J & & \\ L & & & \end{bmatrix} + \alpha_{34} \begin{bmatrix} & & & -L \\ & & J & \\ & J & & \\ L & & & \end{bmatrix}.$$

Now, $M = 0$ means that all its submatrices M_{hk} are zero. Since I, J, K, L are linearly independent, the equations $M_{hk} = 0$ are equivalent to a number of linear equations in the α's, and from these linear equations we can easily see that the α's must all be zero. For example, it is obvious from the equations

$$M_{12} = (\alpha_2+\alpha_{13})K + (\alpha_3-\alpha_{12})L - (\alpha_{46}+\alpha_{57})I - (\alpha_{47}-\alpha_{56})J = 0,$$
$$M_{34} = (\alpha_2-\alpha_{13})I + (\alpha_3+\alpha_{12})J + (-\alpha_{46}+\alpha_{57})K - (\alpha_{47}+\alpha_{56})L = 0$$

that

$$\alpha_2, \quad \alpha_{13}, \quad \alpha_3, \quad \alpha_{12}, \quad \alpha_{46}, \quad \alpha_{57}, \quad \alpha_{47}, \quad \alpha_{56}$$

must all be zero. Thus we have proved that the 28 matrices B_i, B_iB_j are linearly independent.

(ii) Let G be the Lie subgroup of $SO(8)$ generated by the elements $B(\lambda')$, and g its Lie algebra. Then g is a Lie subalgebra of the Lie algebra $o(8)$ of $SO(8)$. We now prove that in fact $g = o(8)$.

From the theory of Lie groups we know that if $t \to f(t)$, where $t \in R$ and $f(t) \in G$, is any curve in G passing through the identity element

$I = f(0)$ of G, then the velocity vector $f'(0)$ of this curve at I is an element of g. Now

$$t \to f_i(t) \equiv (\cos t)I + (\sin t)B_i \qquad (i=1, ..., 7)$$

are obviously curves in G such that $f_i(0) = I$ and $f_i'(0) = B_i$. Therefore, B_i are all elements of g.

Since g is a Lie subalgebra of $o(8)$ and $B_i \in g$, the Lie products $[B_i, B_j] = B_iB_j - B_jB_i = 2B_iB_j$, where $i, j = 1, ..., 7$, and $i < j$, are all in g.

We have thus proved that the 28 linearly independent skew-symmetric matrices, B_i, B_iB_j all belong to $g \subset o(8)$. Since $o(8)$ is the Lie algebra of all skew-symmetric matrices of order 8 and is therefore of dimension 28, g coincides with $o(8)$. This completes the proof of Theorem 2.6.

3. The sphere bundles $S^{2n-1} \to \Phi_n$, $n = 2, 4$, or 8, with fibers on mutually isoclinic n-planes in R^{2n}

In R^{2n}, $n = 2, 4$, or 8, provided with rectangular coordinate system (x, y), let S^{2n-1} be the unit sphere and Φ_n the maximal set of mutually isoclinic n-planes $\{x = 0,\ y = xB(\lambda)\}$ defined in Theorem 1.6. Then with the preparations we have made in § 2, we can now prove

THEOREM 3.1. *In* R^{2n}, $n = 2, 4$, *or* 8, *the n-planes in the maximal set* Φ_n *of mutually isoclinic n-planes slice the unit sphere* S^{2n-1} *into a fiber bundle*

$$\mathscr{I}_n = (S^{2n-1}, \Phi_n, \pi, S^{n-1}, G_n),$$

with base space Φ_n, *projection* π, *fiber* S^{n-1} *and group* G_n, *where* $G_2 = S^1$, $G_4 = S^3$, *and* $G_8 = SO(8)$.

Proof. We prove by exhibiting all the ingredients of a representative coordinate bundle.

(1) The bundle space S^{2n-1} has the equation $xx^T + yy^T = 1$ in R^{2n}.

(2) The base space Φ_n is covered by the two coordinate systems

$$(\Phi_n \backslash \mathbf{O}^{\perp}, \lambda), \quad (\Phi_n \backslash \mathbf{O}, \mu) \tag{2.5}$$

as in the proof of Theorem 2.3, where $\mathbf{O}^{\perp}$ is the n-plane $x = 0$, $\mathbf{O}$ is the n-plane $y = 0$, λ is the parameter in the equation $y = xB(\lambda)$ of an n-plane in $\Phi_n \backslash \mathbf{O}^{\perp}$, and μ is the parameter in the equation $x = yB(\mu)^T$ of an n-plane in $\Phi_n \backslash \mathbf{O}$. Moreover, for an n-plane in the intersection $\Phi_n \backslash \{\mathbf{O}^\perp, \mathbf{O}\}$ of the two coordinate neighborhoods, its two coordinates λ and μ, both nonzero, are related by

$$\mu = \lambda/N(\lambda), \quad \text{or equivalently,} \quad \lambda = \mu/N(\mu). \tag{2.6}$$

(3) The projection $\pi: S^{2n-1} \to \Phi_n$ is the map which sends a point of S^{2n-1} to the unique n-plane in Φ_n containing this point (cf. Theorems 1.2 and 1.4).

To see that π is continuous, we let

$$V_1 = \{(x, y) \in S^{2n-1} : x \neq 0\}, \qquad V_2 = \{(x, y) \in S^{2n-1} : y \neq 0\}.$$

Then $\{V_1, V_2\}$ is an open cover of S^{2n-1}, and $\pi(V_1) = \Phi_n \backslash \mathbf{O}^\perp$, $\pi(V_2) = \Phi_n \backslash \mathbf{O}$. Now by Theorem 2.2, the restriction $\pi \mid V_1$ of π to V_1 sends a point $(u, v) \in V_1 \subset S^{2n-1}$ to the n-plane $y = xB(\lambda)$ in $\Phi_n \backslash \mathbf{O}^\perp$ with coordinate

$$\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) = (vu^T, -vB_1u^T, \ldots, -vB_{n-1}u^T)/(uu)^T.$$

This shows that $\pi \mid V_1$ is continuous. Similarly for $\pi \mid V_2$. Therefore, π is continuous.

(4) The fiber S^{n-1} is the unit sphere $tt^T = 1$ in R^n. Here, $t = [t_1 \ldots t_n]$ is a rectangular coordinate system in R^n.

(5) The group G_n of the bundle is $G_2 = S^1 = SO(2)$, $G_4 = S^3 = SO(4)$, or $G_8 = SO(8)$, for $n = 2, 4$, or 8, respectively.

To see that G_n acts on S^{n-1} effectively, we need only observe that if M is an element of $G_n \subset SO(n)$ such that $tM = t$ for all t with $tt^T = 1$, then $M = I$.

(6) With the coordinate systems (2.5) covering the base space Φ_n as described in (2), the coordinate functions are the maps

$$\phi_1 : (\Phi_n \backslash \mathbf{O}^\perp) \times S^{n-1} \to \pi^{-1}(\Phi_n \backslash \mathbf{O}^\perp),$$
$$\phi_2 : (\Phi_n \backslash \mathbf{O}) \times S^{n-1} \to \pi^{-1}(\Phi_n \backslash \mathbf{O}),$$

defined respectively by

$$\phi_1(\lambda, t) = (x, y) = \frac{(t, tB(\lambda))}{\sqrt{1 + N(\lambda)}}, \tag{3.1}$$

$$\phi_2(\mu, t') = (x', y') = \frac{(t'B(\mu)^T, t')}{\sqrt{1 + N(\mu)}}. \tag{3.2}$$

Here, the λ in $\phi_1(\lambda, t)$ denotes the n-plane in $\Phi_n\backslash\mathbf{O}^\perp$ with equation $y = xB(\lambda)$, and the μ in $\phi_2(\mu, t')$ denotes the n-plane in $\Phi_n\backslash\mathbf{O}$ with equation $x = yB(\mu)^T$.

To justify our definition, we must show that ϕ_1, ϕ_2 are homeomorphisms. Obviously, they are continuous maps. To find ϕ_1^{-1} which sends (x, y) to (λ, t), we first note that $x \neq 0$ (cf. (3)) and the last equation (3.1) is equivalent to

$$y = xB(\lambda), \qquad t = x\sqrt{1 + N(\lambda)}. \tag{3.3}$$

Now, equation $(3.3)_2$ gives t as a continuous function of x and λ, and by Theorem 2.2, equation $(3.3)_1$ determines λ as a continuous function of x and y. Therefore, λ and t are continuous functions of x and y. This proves that ϕ_1^{-1} is well defined and is continuous, and consequently, ϕ_1 is a homeomorphism. Similarly for ϕ_2.

(7) The projection π and the coordinate functions ϕ_1, ϕ_2 as defined in (3) and (6) satisfy the conditions

$$(\pi \circ \phi_1)(\lambda, t) = \lambda, \qquad (\pi \circ \phi_2)(\mu, t') = \mu. \tag{3.4}$$

In fact, from (3.1) and (3.3), we see that the point $(x, y) = \phi_1(\lambda, t)$ of S^{2n-1} lies on the n-plane $y = xB(\lambda)$. Therefore, by (3), $\pi(x, y)$ is the n-plane $y = xB(\lambda)$ in $\Phi_n\backslash\mathbf{O}^\perp$ with coordinate λ. This proves $(3.4)_1$. Similarly for $(3.4)_2$.

(8) Let $\mathbf{B}$ be any fixed n-plane in $\Phi_n\backslash\{\mathbf{O}^\perp, \mathbf{O}\}$ with coordinate λ in $\Phi_n\backslash\mathbf{O}^\perp$ and coordinate μ in $\Phi_n\backslash\mathbf{O}$, and let $\phi_{1,\mathbf{B}}$ and $\phi_{2,\mathbf{B}}$ be the two homeomorphisms $S^{n-1} \to \pi^{-1}(\mathbf{B}) \subset S^{2n-1}$ defined by

$$\phi_{1,\mathbf{B}}(t) = \phi_1(\lambda, t), \qquad \phi_{2,\mathbf{B}}(t') = \phi_2(\mu, t').$$

Then $\phi_{2,\mathbf{B}}^{-1} \circ \phi_{1,\mathbf{B}}$ is a homeomorphism in the fiber S^{n-1}, called a coordinate transformation.

We now show that this coordinate transformation coincides with the action of an element of the group G_n. Suppose that t is any point of S^{n-1} and

$$(\phi_{2,\mathbf{B}}^{-1} \circ \phi_{1,\mathbf{B}})(t) = t' \in S^{n-1}.$$

Then

$$\phi_{1,\mathbf{B}}(t) = \phi_{2,\mathbf{B}}(t'), \qquad \text{i.e.,} \qquad \phi_1(\lambda, t) = \phi_2(\mu, t').$$

Now, by (3.1) and (3.2), this equation is the same as

$$\text{(3.5)} \qquad \frac{(t,\, tB(\lambda))}{\sqrt{1+N(\lambda)}} = \frac{(t'B(\mu)^T,\, t')}{\sqrt{1+N(\mu)}}\,.$$

Since the two coordinates λ, μ of the n-plane $\mathbf{B}$ satisfy the conditions

$$\lambda \neq 0\,, \quad \mu \neq 0\,, \quad B(\lambda)^{-1} = B(\mu)^T\,,$$
$$\mu = \lambda/N(\lambda)\,, \quad \lambda = \mu/N(\mu)\,, \quad N(\lambda)N(\mu) = 1\,,$$

we can easily verify that equation (3.5) is equivalent to

$$\text{(3.6)} \qquad t' = tB(\lambda)/N(\lambda)^{1/2}\,,$$

and this, on putting $\lambda' = \lambda/N(\lambda)^{1/2}$, we can write as

$$\text{(3.6}'\text{)} \qquad t' = tB(\lambda')\,, \quad \text{where} \quad N(\lambda') = 1\,.$$

The transformation (3.6), or equivalently, (3.6′), is then a coordinate transformation in the fiber S^{n-1}. Now, by Theorems 2.4, 2.5 and 2.6, G_n is the subgroup of $SO(n)$ generated by the set of elements $\{B(\lambda'): N(\lambda') = 1\}$ of $SO(n)$. Therefore, the coordinate transformation (3.6′) coincides with the action of an element of G_n.

(9) Finally, we see from (3.6) that the map

$$(\Phi_n \backslash \mathbf{O}^\perp) \cap (\Phi_n \backslash \mathbf{O}) = \Phi_n \backslash \{\mathbf{O}^\perp, \mathbf{O}\} \to G_n\,,$$

defined by $\mathbf{B} \to \phi_{2,\mathbf{B}}^{-1} \circ \phi_{1,\mathbf{B}}$, can be expressed in coordinates as

$$\lambda \to B(\lambda)/N(\lambda)^{1/2}\,.$$

Therefore, it is continuous.

Thus, with the ingredients (1)-(9) exhibited above, we have constructed a representative coordinate bundle of the sphere bundle $\mathscr{S}_n$ in Theorem 3.1.

REMARK 1. In Theorem 2.3, we have shown that Φ_n is diffeomorphic with S^n. Therefore, the three sphere bundles $\mathscr{S}_n$ in Theorem 3.1 are topologically the same as some sphere bundles $S^{2n-1} \to S^n$ by S^{n-1}. In fact, we shall prove in § 5 that they are topologically essentially the same as the three Hopf-Steenrod sphere bundles.

REMARK 2. The coordinate functions ϕ_1 and ϕ_2 which we used in (6) were not accidentally come by. They were obtained in the following way. By definition, the coordinate function

$$\phi_1 : (\Phi_n \backslash \mathbf{O}^\perp) \times S^{n-1} \to \pi^{-1}(\Phi_n \backslash \mathbf{O}^\perp) \subset S^{2n-1}$$

is a homeomorphism sending

$$(\lambda, t) \to (x, y) \in \pi^{-1}(\Phi_n \backslash \mathbf{O}^\perp),$$

so that x, y are some continuous functions of λ and t, and λ, t are some continuous functions of x and y. These functions are not arbitrary, but should satisfy certain conditions. First, they must be such that $(\pi \circ \phi_1)(\lambda, t) = \pi(x, y) = \lambda$ (cf. (7)). Therefore, x and y must be related by

$$y = xB(\lambda). \tag{3.7}$$

Secondly, since $(x, y) \in S^{2n-1}$, we must have $xx^T + yy^T = 1$. Furthermore, because of (3.7) and Theorem 2.1 (i),

$$yy^T = xB(\lambda)\,(xB(\lambda))^T = xx^T N(\lambda).$$

Therefore,

$$xx^T = (1+N(\lambda))^{-1}. \tag{3.8}$$

Finally, since $t \in S^{n-1}$, we must have

$$tt^T = 1. \tag{3.9}$$

Conditions (3.7), (3.8) and (3.9) suggest that the simplest possible choice of the continuous functions x, y of λ and t which define our ϕ_1 are those given in (3.1). Similarly for ϕ_2.

With slight modification, we can prove

THEOREM 3.2. *In R^{2n}, $n = 2, 4,$ or 8, the n-planes in the maximal set Φ_n of mutually isoclinic n-planes slice the space $R^{2n}\backslash O$ into a fiber bundle*

$$\mathscr{I}\mathscr{L}_n = (R^{2n}\backslash O, \Phi_n, \pi, R^n\backslash O, G_n \times \rho_n)$$

with base space Φ_n, projection π, fiber $R^n\backslash O$ and group $G_n \times \rho_n$, where $G_2 = S^1$, $G_4 = S^3$ and $G_8 = SO(8)$, and ρ_n is the group of similitudes in $R^n\backslash O$.

Here, by a similitude in $R^n\backslash O$, we mean a transformation of the form $t \to t\rho$, where ρ is a positive real number.

Proof. The proof is similar to that of Theorem 3.1 but with the following difference. The bundle space is $R^{2n}\backslash O$ provided with a rectangular coordinate system (x, y), and the fiber is $R^n\backslash O$ provided with a rectangular coordinate system t; whereas, the base space Φ_n, with coordinate systems (2.5), is the same as that in the bundle $\mathcal{I}_n$. The projection π is the map which sends a point of $R^{2n}\backslash O$ to the (unique) n-plane in Φ_n containing this point, and the two coordinate functions

$$\phi_1 : (\Phi_n\backslash \mathbf{O}^{\perp}) \times (R^n\backslash O) \to \pi^{-1}(\Phi_n\backslash \mathbf{O}^{\perp}),$$
$$\phi_2 : (\Phi_n\backslash \mathbf{O}) \times (R^n\backslash O) \to \pi^{-1}(\Phi_n\backslash \mathbf{O})$$

are defined respectively by

$$\phi_1(\lambda, t) = (x, y) = (t, tB(\lambda)), \tag{3.10}$$

$$\phi_2(\mu, t') = (x', y') = (t'B(\mu)^T, t'). \tag{3.11}$$

It readily follows from (3.10) and (3.11) that, for any fixed $\mathbf{B} \in \Phi_n\backslash\{\mathbf{O}^{\perp}, \mathbf{O}\}$ with coordinate λ in $\Phi_n\backslash \mathbf{O}^{\perp}$, the coordinate transformation $\phi_{2,\mathbf{B}}^{-1} \circ \phi_{1,\mathbf{B}}$ in the fiber $R^n\backslash O$ sends t to

$$t' = tB(\lambda), \tag{3.12}$$

which, because $\lambda \neq 0$, can be written

$$t' = (tB(\lambda)/N(\lambda)^{1/2})N(\lambda)^{1/2}. \tag{3.12'}$$

Since $B(\lambda)/N(\lambda)^{1/2} \in G_n$ and $t \to tN(\lambda)^{1/2}$ is a similitude in $R^n\backslash O$, (3.12′) shows that the coordinate transformation $t \to t'$ coincides with the action of an element of $G_n \times \rho_n$. Finally, by (3.12), the map $\Phi_n\backslash\{\mathbf{O}^{\perp}, \mathbf{O}\} \to G_n \times \rho_n$ defined by $\mathbf{B} \to \phi_{2,\mathbf{B}}^{-1} \circ \phi_{1,\mathbf{B}}$ can be expressed as $\lambda \to B(\lambda)$, and is therefore continuous.

The relationship between the bundle $\mathcal{I}_n$ in Theorem 3.1 and the bundle $\mathcal{I}\mathcal{L}_n$ in Theorem 3.2 is described in the following

THEOREM 3.3.

(i) *The bundle*

$$\mathcal{I}\mathcal{L}_n = (R^{2n}\backslash O, \Phi_n, \pi, R^n\backslash O, G_n \times \rho_n)$$

is equivalent in $G_n \times \rho_n$ *to the bundle*

$$\mathcal{I}\mathcal{L}'_n = (R^{2n}\backslash O, \Phi_n, \pi, R^n\backslash O, G_n)$$

with group G_n.

(ii) *The bundle*

$$\mathcal{I}_n = (S^{2n-1}, \Phi_n, \pi, S^{n-1}, G_n)$$

is a subbundle of the bundle $\mathcal{I}\mathcal{L}'_n$ *in* (i).

Proof. (i) This is an immediate consequence of a result of Steenrod in [5, p. 56, § 12.6]. In fact, from (6) and (8) in the proof of Theorem 3.1, we easily see that the coordinate functions

$$\phi'_1 : (\Phi_n \backslash \mathbf{O}^\perp) \times (R^n \backslash O) \to \pi^{-1}(\Phi_n \backslash \mathbf{O}^\perp),$$
$$\phi'_2 : (\Phi_n \backslash \mathbf{O}) \times (R^n \backslash O) \to \pi^{-1}(\Phi_n \backslash \mathbf{O})$$

of $\mathscr{I}\mathscr{L}'_n$ can be defined respectively by

$$\phi'_1(\lambda, t) = \frac{(t, tB(\lambda))}{\sqrt{1 + N(\lambda)}},$$

$$\phi'_2(\mu, t') = \frac{(t'B(\mu)^T, t')}{\sqrt{1 + N(\mu)}},$$

and that for any fixed element $\mathbf{B} \in \Phi_n \backslash \{\mathbf{O}^\perp, \mathbf{O}\}$, the coordinate transformation $\phi'^{-1}_{2,\mathbf{B}} \circ \phi'_{1,\mathbf{B}}$ in $R^n \backslash O$ is $t \to t' = tB(\lambda)/N(\lambda)^{1/2}$, and thus it cooincides with the action of an element of G_n.

(ii) Obviously, $S^{n-1} \subset R^n \backslash O$ is invariant under G_n. Therefore, according to a result of Steenrod [5, p. 24, 2nd paragraph], there is a unique subbundle of $\mathscr{I}\mathscr{L}'_n$ with fiber S^{n-1} and the same coordinate neighborhoods and coordinate transformations as $\mathscr{I}\mathscr{L}'_n$. Comparison will show that this subbundle is precisely our $\mathscr{I}_n$.

4. A unified treatment of the three Hopf-Steenrod bundles

In the early 30's, H. Hopf [2, 3], using complex numbers, quaternions, and Cayley numbers, discovered his fiberings of S^{2n-1} by S^{n-1} over S^n, $n = 2, 4, 8$. Later in 1950, N. Steenrod [5, pp. 105-110] used these fiberings of Hopf to construct three sphere bundles, which we here call the *Hopf-Steenrod bundles.* But he did this in a roundabout way. For the two cases $n = 2, 4$, he obtained the bundles $S^3 \to S^2$ and $S^7 \to S^4$ as special cases of a general result on "sphere as a bundle over a projective space". For the case $n = 8$, he obtained the bundle $S^{15} \to S^8$ as a subbundle of a linear bundle which he constructed by using Cayley numbers. This being the case, he did not need to define the coordinate functions for his bundles. Still later in 1952, P.J. Hilton [1, pp. 52-55] showed, in a direct manner, that the Hopf fiberings $S^{2n-1} \to S^n$, $n = 2, 4, 8$, are fiber spaces by exhibiting some sets of coordinate functions. But he did not calculate the coordinate transformations or mention the bundle groups because they were not needed for his purpose.

In this section, we first describe the fiberings of S^{2n-1} by S^{n-1} over S^n, $n = 2, 4, 8$, as Hopf first discovered them, and then, using Hopf's ideas and method and taking into consideration the work of Steenrod and Hilton, we give a unified and explicit formulation of the structures of the three Hopf-Steenrod bundles $S^{2n-1} \to S^n$. In the next section, we shall show how the Hopf-Steenrod bundles are related to the sphere bundles we constructed in § 3.

Let Q_n, $n = 2, 4, 8$, be respectively the (hypercomplex) systems of complex numbers, quaternions and Cayley numbers. (See Appendix 1 for properties of Cayley numbers.) Suppose that I_a, $a = 0, 1, ..., n-1$, are the base elements in Q_n. Then any element X of Q_n can be uniquely expressed as

$$X = \sum_{a=0}^{n-1} x_{a+1} I_a ,$$

where $x_1, ..., x_n$ are real numbers called the components of X. Furthermore, let us define

$$| X | = (\sum_{a=0}^{n-1} x_{a+1}^2)^{1/2}$$

as the length of X. Then we can identify Q_n with the Euclidean n-space R^n by taking the components $(x_1, ..., x_n)$ of an element X in Q_n as the rectangular coordinates of the point X in R^n.

Consider now the space $Q_n \times Q_n$ of ordered pairs (X, Y) of elements of Q_n, and let $x = (x_1, ..., x_n)$ and $y = (x_{n+1}, ..., x_{2n})$ be the components of X and Y. Then we can identify $Q_n \times Q_n$ with R^{2n} by taking $(x, y) = (x_1, ..., x_n; x_{n+1}, ..., x_{2n})$ as rectangular coordinates in R^{2n}. Calling (X, Y) the Q_n-coordinates in R^{2n}, we define a Q_n-*line* in R^{2n} as either the point set $X = 0$, or a set of all the points whose Q_n-coordinates (X, Y) satisfy an equation of the form $Y = CX$, where C is some element of Q_n. We can easily see that the Q_n-lines are n-planes in R^{2n} with the properties that through any point in $R^{2n} \backslash O$, there passes one and only one such n-plane, and that any two such n-planes intersect only at the origin of R^{2n}.

Suppose that S^{2n-1} is the unit sphere $| X |^2 + | Y |^2 = 1$ in R^{2n}. Then it follows from the above that the great $(n-1)$-spheres in which S^{2n-1} is intersected by the Q_n-lines are such that one and only one of them passes through each point of S^{2n-1}, and so they form a fibering of S^{2n-1} by S^{n-1}.

Closely associated with this fibering of S^{2n-1} is a map p from S^{2n-1} onto the n-sphere S^n, defined as follows. First, we regard S^n as R^n closed by the point ∞ at infinity, so that $S^n = R^n \cup \infty = Q_n \cup \infty$. Then p sends each point of S^{2n-1} which lies on a Q_n-line $Y = CX$ to the point $C \in Q_n \subset S^n$, and sends each point of S^{2n-1} which lies on the Q_n-line $X = 0$ to the point $\infty \in S^n$. In other words, the map $p: S^{2n-1} \to S^n$ is defined by

$$p(X, Y) = \begin{cases} YX^{-1} & \text{if} \quad X \neq 0\,, \\ \infty & \text{if} \quad X = 0\,, \end{cases} \tag{4.1}$$

where (X, Y) is any point of S^{2n-1}. It is easy to see that p is a continuous map, and that the inverse image of each point of S^n is one of the great $(n-1)$-spheres in which S^{2n-1} is intersected by the Q_n-lines.

The fibering $S^{2n-1} \to S^n$ by S^{n-1} constructed above is then the famous *Hopf fibering*, and the map p is the *Hopf map* related to it.

We now prove the following theorem which gives a unified and explicit formulation of the three Hopf-Steenrod bundles $S^{2n-1} \to S^n$, $n = 2, 4, 8$.

THEOREM 4.1. *Let* Q_n, $n = 2, 4, 8$, *be respectively the systems of complex numbers, quaternions and Cayley numbers, and let the spaces* Q_n, $Q_n \times Q_n$ *be identified with* R^n, $R^n \times R^n = R^{2n}$, *respectively. Then the set of* Q_n*-lines* $\{X = 0, Y = CX\}$ *in* R^{2n} *slice the unit sphere* S^{2n-1} *in* R^{2n} *into a fiber bundle*

$$\mathscr{H}\mathscr{S}_n = (S^{2n-1}, S^n, p, S^{n-1}, O(n))$$

with base space $S^n = Q_n \cup \infty$, *projection* p, *fiber* S^{n-1} *and group the orthogonal group* $O(n)$.

Proof. We prove by exhibiting the ingredients of a representative coordinate bundle.

(1) The bundle space S^{2n-1} is the unit sphere $|X|^2 + |Y|^2 = 1$ in $R^{2n} = Q_n \times Q_n$.

(2) The base space S^n is identified with $R^n \cup \infty = Q_n \cup \infty$. Therefore, S^n is covered by the two coordinate neighborhoods

$$Q_n, \qquad S^n \backslash O = (Q_n \cup \infty) \backslash O = (Q_n \backslash O) \cup \infty\,,$$

with elements of Q_n and ∞ serving as coordinates.

(3) The projection $p: S^{2n-1} \to S^n$ is the Hopf map defined by (4.1).

(4) The fiber S^{n-1} is the unit sphere $|X| = 1$ in $R^n = Q_n$.

(5) The group $O(n)$ of the bundle acts on S^{n-1} effectively.

(6) Let C, D be elements of Q_n such that $|D| = 1$, so that D represents a point of S^{n-1}. Then the two coordinate functions are the homeomorphisms

$$\psi_1 : Q_n \times S^{n-1} \to p^{-1}(Q_n),$$
$$\psi_2 : (S^n \backslash O) \times S^{n-1} \to p^{-1}(S^n \backslash O),$$

defined respectively by

$$\psi_1(C, D) = \frac{(D, CD)}{\sqrt{1 + |C|^2}}, \tag{4.2}$$

$$\begin{cases} \psi_2(C, D) = \dfrac{(C^{-1}D, D)}{\sqrt{1 + 1/|C|^2}}, & \text{where} \quad C \neq \infty, \\ \\ \psi_2(\infty, D) = (O, D). \end{cases} \tag{4.3}$$

That ψ_1, ψ_2 are indeed homeomorphisms is easy to verify.

(7) It can readily be seen from (4.1), (4.2) and (4.3) that the projection p and the coordinate functions ψ_1, ψ_2 satisfy the conditions:

$$\begin{cases} (p \circ \psi_1)(C, D) = C, \\ (p \circ \psi_2)(C, D) = C \quad \text{if} \quad C \neq \infty, \quad \text{and} \quad (p \circ \psi_2)(\infty, D) = \infty. \end{cases} \tag{4.4}$$

(8) For each fixed point C in the intersection $Q_n \cap (S^n \backslash O) = Q_n \backslash O$ of the two coordinate neighborhoods in the base space S^n, let $\psi_{1,C}$ and $\psi_{2,C}$ be the two homeomorphisms $S^{n-1} \to p^{-1}(C) \subset S^{2n-1}$ defined by

$$\psi_{1,C}(D) = \psi_1(C, D), \quad \psi_{2,C}(D) = \psi_2(C, D),$$

where ψ_1, ψ_2 are the coordinate functions defined in (6). Then, we can easily verify by using (4.2) and (4.3) that the coordinate transformation $\psi_{2,C}^{-1} \circ \psi_{1,C}$ in the fiber S^{n-1} is

$$D \to CD/|C|, \tag{4.5}$$

where D with $|D| = 1$ is a variable point of $S^{n-1} \subset Q_n$. Now since the components of the product CX of any two elements C, X of Q_n are bilinear functions of the components of C, X, the map $X \to CX/|C|$ is a linear transformation in $R^n = Q_n$. It is in fact an orthogonal transformation because $|CX/|C|| = |X|$. Therefore, the coordinate transformation (4.5) coincides with the action of an element of the group $O(n)$.

(9) Finally, from the bilinearity of the product CX, it also follows that the coordinate transformation (4.5) varies continuously with C. Therefore, the

map from $Q_n \cap (S^n \backslash O) = Q_n \backslash O \to O(n)$ defined by $C \to \psi_{2,C}^{-1} \circ \psi_{1,C}$ is continuous.

Thus, with the ingredients (1)-(9) exhibited above, we have constructed a representative coordinate bundle of the bundle $\mathscr{H}\mathscr{S}_n$ in the theorem.

REMARK. The coordinate functions ψ_1 and ψ_2 as given in (4.2) and (4.3) were arrived at as follows. By definition, ψ_1 is a homeomorphism sending

$$(C, D) \in Q_n \times S^{n-1} \to (X, Y) \in p^{-1}(Q_n) \subset S^{2n-1}.$$

Here, X and Y are not arbitrary functions of C, D, but must satisfy certain conditions. First, they must satisfy $(4.4)_1$, so that $(p \circ \psi_1)(C, D) = p(X, Y) = C$. Therefore, by (4.1) X and Y are related by

$$Y = CX. \tag{4.6}$$

Secondly, since (X, Y) is a point of S^{2n-1}, $|X|^2 + |Y|^2 = 1$. Combining this with (4.6), we get

$$|X|^2 = 1/(1+|C|^2). \tag{4.7}$$

Finally, since $D \in S^{n-1} \subset Q_n$,

$$|D| = 1. \tag{4.8}$$

Conditions (4.6), (4.7) and (4.8) suggest that the simplest choice of ψ_1 is (4.2). Similarly, we choose (4.3) as ψ_2 because of conditions $(4.4)_2$.

Similar to Theorem 4.1, we have

THEOREM 4.2. *In R^{2n}, $n = 2, 4,$ or 8, the Q_n-lines slice $R^{2n} \backslash O$ into a fiber bundle*

$$\mathscr{S}\mathscr{L}_n = (R^{2n} \backslash O, S^n, p, Q_n \backslash O, GL(n, R))$$

with base space $S^n = Q_n \cup \infty$, projection p, fiber $Q_n \backslash O$, and group the general linear group $GL(n, R)$.

Proof. The proof is similar to that of Theorem 4.1, but with the following difference. The projection is the map $p: R^{2n} \backslash O \to S^n$ defined by

$$p(X, Y) = \begin{cases} YX^{-1} & \text{if} \quad X \neq 0, \\ \infty & \text{if} \quad X = 0; \end{cases} \tag{4.9}$$

and the two coordination functions

$$\psi_1 : Q_n \times (Q_n \backslash O) \to p^{-1}(Q_n),$$
$$\psi_2 : (S^n \backslash O) \times (Q_n \backslash O) \to p^{-1}(S^n \backslash O)$$

are defined respectively by

$$\psi_1(C, D) = (D, CD), \tag{4.10}$$

$$\begin{cases} \psi_2(C, D) = (C^{-1}D, D), & \text{where} \quad C \neq \infty, \\ \psi_2(\infty, D) = (O, D). \end{cases} \tag{4.11}$$

The coordinate transformations $\psi_{2,C}^{-1} \circ \psi_{1,C}$, where $C \in Q_n \cap (S^n \backslash O) = Q_n \backslash O$, are the linear maps $D \to CD$ in the fiber $Q_n \backslash O$.

The relationship between the bundles $\mathscr{H}\mathscr{S}_n$ and $\mathscr{S}\mathscr{L}_n$ is described in the following theorem, the proof of which is similar to that of Theorem 3.3.

THEOREM 4.3.

(i) *The bundle*

$$\mathscr{S}\mathscr{L}_n = (R^{2n} \backslash O, S^n, p, Q_n \backslash O, GL(n, R))$$

is equivalent in $GL(n, R)$ *to the bundle*

$$\mathscr{S}\mathscr{L}'_n = (R^{2n} \backslash O, S^n, p, Q_n \backslash O, O(n))$$

with group $O(n)$.

(ii) *The bundle*

$$\mathscr{H}\mathscr{S}_n = (S^{2n-1}, S^n, p, S^{n-1}, O(n))$$

is a subbundle of the bundle $\mathscr{S}\mathscr{L}'_n$.

Let us now explain how the bundle $\mathscr{H}\mathscr{S}_n$ given in Theorem 4.1 is a unified formulation of the sphere bundles $S^{2n-1} \to S^n$, $n = 2, 4, 8$ constructed by N. Steenrod using the Hopf fiberings, and how our construction incorporates the work of P. J. Hilton.

(a) Comparison of the ingredients of the sphere bundle $\mathscr{H}\mathscr{S}_8$ in Theorem 4.1 with those of the fiber space $S^{15} \to S^8$ of Hilton [1, p. 54] shows that they have the same projection (4.1) and coordinate functions (4.2) and (4.3).

(b) Suppose that in the construction of the sphere bundle $\mathscr{H}\mathscr{S}_n$ in Theorem 4.1, we use the "Q_n-lines" $X = CY$ instead of the Q_n-lines $Y = CX$ in defining the projection $p : S^{2n-1} \to S^n$. Then we can obtain another sphere bundle $S^{2n-1} \to S^n$ by using the ingredients of $\mathscr{H}\mathscr{S}_n$ but interchanging the roles of X and Y, i.e., by replacing

(i) the projection (4.1) by

$$p(X, Y) = \begin{cases} XY^{-1} & \text{if} \quad Y \neq 0, \\ \infty & \text{if} \quad Y = 0, \end{cases} \tag{4.1'}$$

and

(ii) the coordinate functions (4.2) and (4.3) by

$$\psi_1(C, D) = \frac{(CD, D)}{\sqrt{1 + |C|^2}}, \tag{4.2'}$$

$$\begin{cases} \psi_2(C, D) = \dfrac{(D, C^{-1}D)}{\sqrt{1 + 1/|C|^2}}, & \text{where} \quad C \neq \infty, \\ \psi_2(\infty, D) = (D, O). \end{cases} \tag{4.3'}$$

For $n = 2$, the X, Y, C and D (with $|D|=1$) are all complex numbers. On putting $X = z_1$, $Y = z_2$, $C = \mu$ and $D = e^{i\theta}$, we can see immediately that the projection (4.1′) and the coordinate functions (4.2′) and (4.3′) are exactly those used by Hilton [1, p. 51] to prove that the Hopf fibering $S^3 \to S^2$ has a fiber space structure.

(c) Suppose that in the construction of the linear bundle $\mathscr{S}\mathscr{L}_n$ in Theorem 4.2, we use the "Q_n-lines" $X = CY$ instead of the Q_n-lines $Y = CX$ in defining the projection $p: R^{2n}\backslash O \to S^n$. Then we can obtain another linear bundle by using the ingredients of $\mathscr{S}\mathscr{L}_n$, but interchanging the roles of X and Y, i.e., by replacing

(i) the projection (4.9) by

$$p(X, Y) = \begin{cases} XY^{-1} & \text{if} \quad Y \neq 0, \\ \infty & \text{if} \quad Y = 0, \end{cases} \tag{4.9'}$$

and

(ii) the coordinate functions (4.10) and (4.11) by

$$\psi_1(C, D) = (CD, D), \tag{4.10'}$$

$$\begin{cases} \psi_2(C, D) = (D, C^{-1}D), & \text{where} \quad C \neq \infty, \\ \psi_2(\infty, D) = (D, O). \end{cases} \tag{4.11'}$$

For $n = 8$, the X, Y, C and D are Cayley numbers. On putting $X = c$, $Y = d$, $C = x$ and $D = y$, we can see immediately that the projection (4.9′) and the coordinate functions (4.10′) and (4.11′) are exactly those of the linear bundle $\mathscr{B}$ constructed by N. Steenrod in [5, pp. 109-110]. Therefore, this linear bundle $\mathscr{B}$ of Steenrod and the linear bundle $\mathscr{S}\mathscr{L}_8$ in Theorem 4.2 are two slightly different representations of the same bundle.

5. Comparison of our bundles with the Hopf-Steenrod bundles

In § 3, we constructed the sphere bundles

$$\mathscr{I}_n = (S^{2n-1}, \Phi_n, \pi, S^{n-1}, G_n), \qquad n = 2, 4, 8,$$

with fibers lying on mutually isoclinic n-planes in R^{2n}. In § 4, we gave a unified treatment of the classical Hopf-Steenrod sphere bundles

$$\mathscr{H}\mathscr{S}_n = \big(S^{2n-1}, S^n, p, S^{n-1}, O(n)\big), \qquad n = 2, 4, 8,$$

using, as N. Steenrod did, the Hopf map and the hypercomplex systems of complex numbers, quaternions and Cayley numbers. In this section we shall prove that (i) the Hopf fibering $S^{2n-1} \to S^n$ and our maximal set of mutually isoclinic n-planes in R^{2n} are equivalent concepts (Theorems 5.1 and 5.2), and (ii) the representative coordinate bundles constructed in § 3 and § 4 for the bundles $\mathscr{I}_n$ and $\mathscr{H}\mathscr{S}_n$ are topologically essentially the same (Theorem 5.3). For convenience, the theorems will be stated and proofs given for the case $n = 8$ only. Similar theorems hold for the cases $n = 2, 4$, and their proofs follow the same line and are simpler.

THEOREM 5.1. *For $n = 8$, let us identify the space Q_8 of Cayley numbers with R^8 by regarding the Cayley number*

$$X \equiv (x_1+x_2i+x_3j+x_4k\,,\ x_5+x_6i+x_7j+x_8k)$$

as the point in R^8 with rectangular coordinates $(x_1, ..., x_8)$, and the space $Q_8 \times Q_8$ of ordered pairs of Cayley numbers with $R^8 \times R^8 = R^{16}$ by regarding the ordered pair

$$\begin{aligned}(X, Y) \equiv \big((x_1+x_2i+x_3j+x_4k\,,\ &x_5+x_6i+x_7j+x_8k)\,,\\ (x_9+x_{10}i+x_{11}j+x_{12}k\,,\ &x_{13}+x_{14}i+x_{15}j+x_{16}k)\big)\end{aligned}$$

as the point in R^{16} with rectangular coordinates $(x_1, ..., x_8; x_9, ..., x_{16})$. Then, written in terms of $x = [x_1 \ldots x_8]$ and $y = [x_9 \ldots x_{16}]$,

(i) *the equation* $X = 0$ *becomes* $x = 0$;

(ii) *the equation* $Y = CX$ *becomes* $y = xB(\lambda)$, *where* $B(\lambda)$ *is the* 8×8 *matrix in Theorem 1.6 (iii) and* $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_7)$ *is related to* C *by*

$$C = (\lambda_0+\lambda_1 i+\lambda_2 j+\lambda_3 k\,,\ \lambda_4+\lambda_5 i+\lambda_6 j+\lambda_7 k)\,. \tag{5.1}$$

Proof. Since (i) is obvious, we shall prove (ii) only. Let $X = (p, q)$, $Y = (r, s)$ and $C = (a, b)$. Then the equation $Y = CX$ is

$$(r, s) = (a, b)\,(p, q) = (ap-q^*b,\ qa+bp^*)\,,$$

i.e.,

$$\begin{aligned}
&(x_9+x_{10}i+x_{11}j+x_{12}k\,,\ x_{13}+x_{14}i+x_{15}j+x_{16}k)\\
&\quad = (\lambda_0+\lambda_1 i+\lambda_2 j+\lambda_3 k\,,\ \lambda_4+\lambda_5 i+\lambda_6 j+\lambda_7 k)\\
&\qquad\quad \times (x_1+x_2 i+x_3 j+x_4 k\,,\ x_5+x_6 i+x_7 j+x_8 k)\\
&\quad = \big((\lambda_0+\lambda_1 i+\lambda_2 j+\lambda_3 k)\,(x_1+x_2 i+x_3 j+x_4 k)\\
&\qquad\quad - (x_5-x_6 i-x_7 j-x_8 k)\,(\lambda_4+\lambda_5 i+\lambda_6 j+\lambda_7 k)\,,\\
&\qquad (x_5+x_6 i+x_7 j+x_8 k)\,(\lambda_0+\lambda_1 i+\lambda_2 j+\lambda_3 k)\\
&\qquad\quad + (\lambda_4+\lambda_5 i+\lambda_6 j+\lambda_7 k)\,(x_1-x_2 i-x_3 j-x_4 k)\big)\\
&\quad = \big((\lambda_0 x_1-\lambda_1 x_2-\lambda_2 x_3-\lambda_3 x_4) - (x_5\lambda_4+x_6\lambda_5+x_7\lambda_6+x_8\lambda_7)\\
&\qquad\quad + (\lambda_0 x_2+\lambda_1 x_1+\lambda_2 x_4-\lambda_3 x_3)i - (x_5\lambda_5-x_6\lambda_4-x_7\lambda_7+x_8\lambda_6)i\\
&\qquad\quad + (\lambda_0 x_3-\lambda_1 x_4+\lambda_2 x_1+\lambda_3 x_2)j - (x_5\lambda_6+x_6\lambda_7-x_7\lambda_4-x_8\lambda_5)j\\
&\qquad\quad + (\lambda_0 x_4+\lambda_1 x_3-\lambda_2 x_2+\lambda_3 x_1)k - (x_5\lambda_7-x_6\lambda_6+x_7\lambda_5-x_8\lambda_4)k\,,\\
&\qquad (x_5\lambda_0-x_6\lambda_1-x_7\lambda_2-x_8\lambda_3) + (\lambda_4 x_1+\lambda_5 x_2+\lambda_6 x_3+\lambda_7 x_4)\\
&\qquad\quad + (x_5\lambda_1+x_6\lambda_0+x_7\lambda_3-x_8\lambda_2)i + (-\lambda_4 x_2+\lambda_5 x_1-\lambda_6 x_4+\lambda_7 x_3)i\\
&\qquad\quad + (x_5\lambda_2-x_6\lambda_3+x_7\lambda_0+x_8\lambda_1)j + (-\lambda_4 x_3+\lambda_5 x_4+\lambda_6 x_1-\lambda_7 x_2)j\\
&\qquad\quad + (x_5\lambda_3+x_6\lambda_2-x_7\lambda_1+x_8\lambda_0)k + (-\lambda_4 x_4-\lambda_5 x_3+\lambda_6 x_2+\lambda_7 x_1)k\big)\,,
\end{aligned}$$

which is easily seen to be equivalent to

$$[x_9 \ldots x_{16}] = [x_1 \ldots x_8]\begin{bmatrix}
\lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7\\
-\lambda_1 & \lambda_0 & \lambda_3 & -\lambda_2 & \lambda_5 & -\lambda_4 & -\lambda_7 & \lambda_6\\
-\lambda_2 & -\lambda_3 & \lambda_0 & \lambda_1 & \lambda_6 & \lambda_7 & -\lambda_4 & -\lambda_5\\
-\lambda_3 & \lambda_2 & -\lambda_1 & \lambda_0 & \lambda_7 & -\lambda_6 & \lambda_5 & -\lambda_4\\
-\lambda_4 & -\lambda_5 & -\lambda_6 & -\lambda_7 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3\\
-\lambda_5 & \lambda_4 & -\lambda_7 & \lambda_6 & -\lambda_1 & \lambda_0 & -\lambda_3 & \lambda_2\\
-\lambda_6 & \lambda_7 & \lambda_4 & -\lambda_5 & -\lambda_2 & \lambda_3 & \lambda_0 & -\lambda_1\\
-\lambda_7 & -\lambda_6 & \lambda_5 & \lambda_4 & -\lambda_3 & -\lambda_2 & \lambda_1 & \lambda_0
\end{bmatrix},$$

i.e. to $y = xB(\lambda)$.

An immediate consequence of Theorem 5.1 and Theorem 1.6 (iii) is the following

THEOREM 5.2. *The Hopf fibering* $S^{15} \to S^8$ *and our maximal set of mutually isoclinic* 8*-planes in* R^{16} *are equivalent concepts. More precisely, under the identification of* $Q_8 \times Q_8$ *with* R^{16} *described in Theorem 5.1, the set of* Q_8*-lines* $\{X = 0, Y = CX\}$ *in* $Q_8 \times Q_8$ *corresponds to the maximal set* Φ_8 *of mutually isoclinic* 8*-planes* $\{x = 0, y = xB(\lambda)\}$ *in* R^{16}.

(In Appendix 2, we shall prove, by working directly with Cayley numbers, that the Q_8-lines, regarded as 8-planes in R^{16}, are mutually isoclinic 8-planes.)

We are now ready to prove our main

THEOREM 5.3. *The representative coordinate bundles constructed in* § 4 *and* § 3 *for the sphere bundles* $\mathscr{HS}_8$ *and* $\mathscr{I}_8$ *are topologically essentially the same, with only the group* $SO(8)$ *in* $\mathscr{I}_8$ *replacing the group* $O(8)$ *in* $\mathscr{HS}_8$.

Proof. We first identify the bundle space, fiber and base space in $\mathscr{HS}_8$ with those in $\mathscr{I}_8$, and then show that, under this identification, the projection, coordinate functions and coordinate transformations in $\mathscr{HS}_8$ correspond to those in $\mathscr{I}_8$.

(a) The bundle spaces and the fibers.

The bundle space in $\mathscr{HS}_8$ is the unit sphere $S^{15}: |X|^2 + |Y|^2 = 1$ in $Q_8 \times Q_8$, and that in $\mathscr{I}_8$ is the unit sphere $S^{15}: xx^T + yy^T = 1$ in R^{16}. The fiber in $\mathscr{HS}_8$ is the unit sphere $S^7: |X| = 1$ in Q_8, and that in $\mathscr{I}_8$ is the unit sphere $S^7: tt^T = 1$ in R^8. Let us identify these two S^{15}'s and two S^7's by identifying Q_8 with R^8 and $Q_8 \times Q_8$ with $R^{16} = R_8 \times R_8$ as in Theorem 5.1.

(b) The base spaces.

By definition, the base space in $\mathscr{HS}_8$ is $S^8 \equiv Q_8 \cup \infty$ covered by the open sets

$$\{Q_8, S^8 \backslash O = (Q_8 \backslash O) \cup \infty\},$$

with the Cayley numbers and ∞ serving as coordinates. On the other hand, the base space in $\mathscr{I}_8$ is Φ_8 covered by the open sets

$$\{\Phi_8 \backslash \mathbf{O}^{\perp}, \Phi_8 \backslash \mathbf{O}\}$$

such that an 8-plane $y = xB(\lambda)$ in $\Phi_8\backslash\mathbf{O}^\perp$ has the coordinate λ and an 8-plane $x = yB(\mu)^T$ in $\Phi_8\backslash\mathbf{O}$ has the coordinate μ.

Now $Q_8 \cup \infty$ can be regarded as the image of the unit sphere S^8 in R^9 under the stereographic projection from the north pole of S^8 onto the equator 8-plane, and (by Theorem 2.3) there is a homeomorphism from Φ_8 to S^8 which sends the 8-plane $y = xB(\lambda)$ in Φ_8 to the point of S^8 whose stereographic projection is the point λ on the equator 8-plane. Therefore, we can identify $Q_8 \cup \infty$ with Φ_8 by means of a homeomorphism defined as follows.

Let $j_1 : Q_8 \to \Phi_8\backslash\mathbf{O}^\perp$ be the map which sends the point

$$C = (\lambda_0+\lambda_1 i+\lambda_2 j+\lambda_3 k\,,\ \lambda_4+\lambda_5 i+\lambda_6 j+\lambda_7 k)$$

in Q_8 to the 8-plane $y = xB(\lambda)$ in $\Phi_8\backslash\mathbf{O}^\perp$ with coordinate

$$\lambda = (\lambda_0, \lambda_1, ..., \lambda_7)\,;$$

and $j_2 : S^8\backslash O = (Q_8\backslash O) \cup \infty \to \Phi_8\backslash\mathbf{O}$ the map which sends the point

$$C = (\mu_0+\mu_1 i+\mu_2 j+\mu_3 k\,,\ \mu_4+\mu_5 i+\mu_6 j+\mu_7 k)/N(\mu)$$

in $Q_8\backslash O \subset S^8\backslash O$ to the 8-plane $x = yB(\mu)^T$ in $\Phi_8\backslash\mathbf{O}$ with coordinate

$$\mu = (\mu_0, \mu_1, ..., \mu_7)\,,$$

and the point $\infty \in S^8\backslash O$ to the 8-plane $\mathbf{O}^\perp : x = 0$ in $\Phi_8\backslash\mathbf{O}$ with coordinate $\mu = 0$. Then it follows easily from Theorem 2.3 and its proof that the map $j_1 \cup j_2$ is a homeomorphism from the base space $S^8 = Q_8 \cup \infty$ in $\mathscr{H}\mathscr{S}_8$ to the base space Φ_8 in $\mathscr{I}_8$.

Let us identify these two base spaces by means of the homeomorphism $j_1 \cup j_2$.

(c) The projections.

We now prove that, under the identification defined in (a) and (b) above, the projection p in $\mathscr{H}\mathscr{S}_8$ coincides with the projection π in $\mathscr{I}_8$. Suppose that P is a point of S^{15} lying on the Q_8-line $Y = CX$. Then $p(P) = C \in Q_8 \subset S^8$. Now by Theorem 5.1, this point P lies on the 8-plane $y = xB(\lambda)$ in R^{16}, where $\lambda = (\lambda_0, \lambda_1, ..., \lambda_7)$ is related to C by (5.1). Therefore, under the identification of S^8 and Φ_8 defined in (b), $p(P) = \pi(P)$. Suppose now that P is a point of S^{15} lying on the Q_8-line $X = 0$. Then $p(P) = \infty \in S^8$. By Theorem 5.1, this point P lies on the 8-plane $\mathbf{O}^\perp : x = 0$ in R^{16}. Therefore, $\pi(P)$ is the 8-plane $\mathbf{O}^\perp$ in Φ_8. Since the point $\infty \in S^8$ corresponds to the 8-plane $\mathbf{O}^{\perp}$ in Φ_8 under the identification defined in (b), $p(P) = \pi(P)$. Hence our proof that p and π coincide is complete.

(d) The coordinate functions.

Consider first the coordinate functions

$$\psi_1 : Q_8 \times S^7 \to p^{-1}(Q_8) \quad \text{and} \quad \phi_1 : (\Phi_8 \backslash \mathbf{O}^{\perp}) \times S^7 \to \pi^{-1}(\Phi_8 \backslash \mathbf{O}^{\perp})$$

in $\mathscr{H}\mathscr{S}_8$ and $\mathscr{I}_8$, given by (4.2) and (3.1) respectively. Suppose that under the identification defined in (b) and (a), the element $(C, D) \in Q_8 \times S^7$ corresponds to the element $(\lambda, t) \in (\Phi_8 \backslash \mathbf{O}) \times S^7$. Then C and λ are related by

$$C = (\lambda_0 + \lambda_1 i + \lambda_2 j + \lambda_3 k\,,\ \lambda_4 + \lambda_5 i + \lambda_6 j + \lambda_7 k)\,, \tag{5.1}$$

and D and t by

$$D = (t_1 + t_2 i + t_3 j + t_4 k\,,\ t_5 + t_6 i + t_7 j + t_8 k)\,. \tag{5.2}$$

Now since $D \in S^7 \subset Q_8$ and $t \in S^7 \subset R^8$, we have $|D| = 1$ and $tt^T = 1$, and, by Theorem 5.1, the product CD corresponds to $tB(\lambda)$. Therefore,

$$|C|^2 = |C|^2 |D|^2 = |CD|^2 = tB(\lambda)\left(tB(\lambda)\right)^T = tB(\lambda)B(\lambda)^T t^T = N(\lambda)\,,$$

and

$$\psi_1(C, D) = \frac{(D, CD)}{\sqrt{1 + |C|^2}} \quad \text{corresponds to} \quad \phi_1(\lambda, t) = \frac{(t, tB(\lambda))}{\sqrt{1 + N(\lambda)}}\,.$$

Next, consider the coordinate functions

$$\psi_2 : (S^8 \backslash O) \times S^7 \to p^{-1}(S^8 \backslash O) \quad \text{and} \quad \phi_2 : (\Phi_8 \backslash \mathbf{O}) \times S^7 \to \pi^{-1}(\Phi_8 \backslash \mathbf{O})$$

in $\mathscr{H}\mathscr{S}_8$ and $\mathscr{I}_8$, given by (4.3) and (3.2) respectively. Suppose that under the identification defined in (b) and (a), the element $(C, D) \in (S^8 \backslash O) \times S^7$ corresponds to the element $(\mu, t') \in (\Phi_8 \backslash \mathbf{O}) \times S^7$. Then, C and μ are related by

$$C = (\mu_0 + \mu_1 i + \mu_2 j + \mu_3 k\,,\ \mu_4 + \mu_5 i + \mu_6 j + \mu_7 k)/N(\mu)\,, \tag{5.3}$$

and D and t' by

$$D = (t'_1 + t'_2 i + t'_3 j + t'_4 k\,,\ t'_5 + t'_6 i + t'_7 j + t'_8 k)\,. \tag{5.4}$$

Since (5.3) implies that $|C|^2 = N(\mu)^{-1}$, (5.3) is equivalent to

$$C^{-1} = C^*/|C|^2 = (\mu_0 - \mu_1 i - \mu_2 j - \mu_3 k\,,\ -\mu_4 - \mu_5 i - \mu_6 j - \mu_7 k)\,. \tag{5.3'}$$

Therefore, by Theorem 5.1, $C^{-1}D$ corresponds to

$$t'(\mu_0 - \mu_1 B_1 - \dots - \mu_7 B_7) = t'B(\mu)^T\,.$$

Hence, it follows from the above that

$$\psi_2(C, D) = \frac{(C^{-1}D, D)}{\sqrt{1 + 1/|C|^2}} \quad \text{corresponds to } \phi_2(\mu, t') = \frac{(t'B(\mu)^T, t')}{\sqrt{1 + N(\mu)}}.$$

To complete the proof that ψ_2 and ϕ_2 correspond, we need only observe that under the identification defined in (a) and (b), the point $\infty \in S^8\backslash O$ corresponds to the 8-plane $\mathbf{O}^\perp$: $x = 0$ in $\Phi_8\backslash\mathbf{O}$, and the point

$$\psi_2(\infty, D) = (O, D) \in p^{-1}(S^8\backslash O)$$

of S^{15} coincides with the point

$$\phi_2(O, t') = (O, t') \in \pi^{-1}(\Phi_8\backslash\mathbf{O}).$$

(e) The coordinate transformations and the bundle groups.

Suppose that in $\mathscr{H}\mathscr{S}_8$, C is any point in the intersection $Q_8 \cap (S^8\backslash O) = Q_8\backslash O$ of the two coordinate neighborhoods in the base space S^8, and $D \in Q_8$ with $|D| = 1$ is a variable point of the fiber S^7. Then the coordinate transformations in the fiber S^7 are $D \to CD/|C|$ (cf. proof of Theorem 4.1). Now let λ be the point in the intersection $(\Phi_8\backslash\mathbf{O}^\perp) \cap (\Phi_8\backslash\mathbf{O}) = \Phi_8\backslash\{\mathbf{O}^\perp, \mathbf{O}\}$ of two coordinate neighborhoods in the base space Φ_8 in $\mathscr{I}_8$ corresponding to the point C under the identification defined in (b) above, and $t \in R^8$ with $tt^T = 1$ the point of the fiber S^7 in $\mathscr{I}_8$ corresponding to the point D under the identification defined in (a). Then C and λ are related by (5.1), and D and t are related by (5.2). Since (5.1) implies that $|C|^2 = N(\lambda)$ and since by Theorem 5.1. CD corresponds to $tB(\lambda)$, the coordinate transformations $D \to CD/|C|$ in $\mathscr{H}\mathscr{S}_8$ correspond to the coordinate transformations $t \to tB(\lambda)/N(\lambda)^{1/2}$ in $\mathscr{I}_8$.

Since the bundle group of a coordinate bundle may be taken as the group generated by the coordinate transformations in the fiber, or any effective transformation group of the fiber containing this group, it follows from the above that the bundle groups in $\mathscr{H}\mathscr{S}_8$ and $\mathscr{I}_8$ are the same. Now, the bundle group in $\mathscr{H}\mathscr{S}_8$ as originally given by N. Steenrod is $O(8)$; whereas, we have shown in § 3 that the bundle group in $\mathscr{I}_8$ is $G_8 = SO(8)$ and, moreover, it cannot be replaced by any smaller subgroup of $SO(8)$.

The proof of Theorem 5.3 is now complete.

Let us now consider the cases $n = 2$ and 4. By using the results similar to those in Theorem 5.1 for the case $n = 8$, we can prove, as in (e) above, that the coordinate transformations $D \to CD/|C|$ in $\mathscr{H}\mathscr{S}_4$

correspond to the coordinate transformations $t \to tB(\lambda)/N(\lambda)^{1/2}$ in $\mathscr{I}_4$, where $B(\lambda)$ are the matrices given in (1.7) in Theorem 1.6. By Theorem 2.5, the elements $B(\lambda)/N(\lambda)^{1/2}$ of $SO(4)$ form a subgroup isomorphic with S^3. Therefore, the bundle group $O(4)$ in $\mathscr{H}\mathscr{S}_4$ can be replaced by S^3. Similarly, the bundle group $O(2)$ in $\mathscr{H}\mathscr{S}_2$ can be replaced by S^1. With these observations, we can now prove the following theorem by proceeding as in the proof of Theorem 5.3.

THEOREM 5.4. *The representative coordinate bundles constructed in* § 4 *for the sphere bundles* $\mathscr{H}\mathscr{S}_2$ *and* $\mathscr{H}\mathscr{S}_4$, *with bundle groups* S^1 *and* S^3 *respectively, are topologically the same as the representative coordinate bundles constructed in* § 3 *for the sphere bundles* $\mathscr{I}_2$ *and* $\mathscr{I}_4$, *respectively.*

Finally, we remark that representative coordinate bundles of the bundles $\mathscr{S}\mathscr{L}_n$ in Theorem 4.2 are topologically essentially the same as the representative coordinate bundles of the bundles $\mathscr{I}\mathscr{L}_n$ in Theorem 3.2.

APPENDIX 1. THE CAYLEY NUMBERS

The Cayley numbers, denoted by X, Y, Z, W, etc. are ordered pairs (q_1, q_2) of quaternions subject to the rules and having the properties listed below. The set of all Cayley numbers, therefore, forms a (non-commutative and non-associative) real division algebra. No proof of the properties will be given as they can all be checked by direct computations.

(1) The *addition* is defined by

$$(q_1, q_2) + (q'_1, q'_2) = (q_1+q'_1, q_2+q'_2).$$

The *zero* is $O = (O, O)$.

(2) The *multiplication* is defined by

$$(q_1, q_2)(q'_1, q'_2) = (q_1q'_1 - q'^*_2 q_2, q'_2 q_1 + q_2 q'^*_1),$$

where q'^*_1, q'^*_2 are respectively the conjugates of (the quaternions) q'_1, q'_2. The (two-sided) *unit* is $1 \equiv (1, 0)$.

(3) Multiplication is

(i) distributive with respect to addition, i.e.,

$$(X+Y)W = XW + YW, \quad W(X+Y) = WX + WY;$$

(ii) not commutative, i.e., generally, $XY \neq YX$ (but see (4) (iv) below);

(iii) not associative, i.e., generally, $(XY)W \neq X(YW)$ (but see (7) below).

(4) The *real part* of $X \equiv (q_1, q_2)$ is $\operatorname{Re} X = (\operatorname{Re} q_1, O) \equiv \operatorname{Re} q_1$. X is said to be *real* if $X = \operatorname{Re} X$; i.e., (q_1, q_2) is real iff q_1 is real and $q_2 = 0$.

(i) $\operatorname{Re}(X+Y) = \operatorname{Re}(X) + \operatorname{Re}(Y)$.

(ii) $\operatorname{Re}(XY) = \operatorname{Re}(YX)$.

(iii) $\operatorname{Re}(CX) = 0$ for all X implies that $C = 0$.

(iv) $CX = XC$ for all X iff C is real. In this case, $C = (c_1, 0)$, where $c_1 =$ real, and $CX = (c_1q_1, c_1q_2) = XC$.

(5) The *conjugate* of $X \equiv (q_1, q_2)$ is $X^* = (q_1^*, -q_2)$.

(i) $(X+Y)^* = X^* + Y^*$,

(ii) $(XY)^* = Y^*X^*$.

(iii) $X^* = X$ iff X is real.

(6) The *norm* of X is the non-negative real number $N(X) \equiv XX^*$, which is also equal to X^*X. The *length* of X is the non-negative real number $|X| \equiv N(X)^{1/2} = (XX^*)^{1/2}$.

(i) $N(X) = 0$ iff $X = 0$.

(ii) If $X \neq 0$, then $X^{-1} \equiv X^*/N(X)$ is a right and left inverse of X.

(iii) $N(XY) = N(X)N(Y)$. It follows from this that $XY = 0$ iff $X = 0$ or $Y = 0$.

(7) Though multiplication is generally non-associative,

(i) $(XY)Y^* = X(YY^*)$.

(ii) If $Y \neq 0$, then $(XY)Y^{-1} = X = Y^{-1}(YX)$.

(iii) $\operatorname{Re}((XY)W) = \operatorname{Re}(X(YW))$.

Appendix 2. The Hopf fibering and mutually isoclinic planes

At the beginning of § 4, we described how H. Hopf obtained his fibering of S^{2n-1} by S^{n-1} over $S^n, n = 2, 4,$ or 8, by intersecting the unit sphere S^{2n-1} in $R^{2n} = Q_n \times Q_n$ with the Q_n-lines $Y = CX$ and $X = 0$. In Theorem 5.2, we proved that the Hopf fibering and maximal set of mutually isoclinic n-planes in R^{2n} are equivalent concepts. Here we prove, directly, the

THEOREM A2.1. *The set of Q_n-lines $\{Y = CX, X = 0\}$ in $Q_n \times Q_n$, when viewed as n-planes in R^{2n}, are mutually isoclinic n-planes.*

Proof. We shall prove the theorem for the case $n = 8$ only. The proof for the cases $n = 2, 4$ follows the same line and is simpler.

Some preliminaries are necessary. Suppose that under the identification of $Q_8 \times Q_8$ with R^{16} as in Theorem 5.1, the elements (X, Y), (X', Y') of $Q_8 \times Q_8$ become the vectors (X, Y), (X', Y') in R^{16} with respectively the components $(x_1, \ldots, x_{16})$, $(x'_1, \ldots, x'_{16})$. Then it can easily be verified that the inner product of the two vectors (X, Y) and (X', Y') is

$$<(X, Y), (X', Y')> \equiv \sum_{i=1}^{16} x_i x'_i = \mathrm{Re}\,(XX'^* + YY'^*)\,.$$

It follows from this that the length of the vector (X, Y) is

$$|(X, Y)| = <(X, Y), (X, Y)>^{1/2} = (XX^* + YY^*)^{1/2}\,,$$

and that the two vectors (X, Y) and (X', Y') are orthogonal if and only if $\mathrm{Re}\,(XX'^* + YY'^*) = 0$.

We can now prove our theorem by showing that in R^{16}, the 8-plane $\mathbf{A}: Y = AX$ is isoclinic with the 8-planes $\mathbf{B}: Y = BX$ and $\mathbf{O}^{\perp}: X = 0$.

Let $(T, BT) \in \mathbf{B}$ be the projection of any nonzero vector $(X, AX) \in \mathbf{A}$ on $\mathbf{B}$. Then the vector $(X-T, AX-BT)$ is orthogonal to $\mathbf{B}$, i.e., it is orthogonal to all the vectors $(W, BW) \in \mathbf{B}$, where W is an arbitrary Cayley number. Therefore,

$$\text{(A.1)} \qquad \mathrm{Re}\,\{(X-T)W^* + (AX-BT)(BW)^*\} = 0 \quad \text{for all} \quad W \in Q_8\,.$$

Since, by (4) (ii) and (7) (iii) in Appendix 1, the terms inside the brackets in Re { } are commutative and associative, the left-hand side of (A.1) is equal to

$$\begin{aligned} &\mathrm{Re}\,\{(X-T)W^* + [(AX-BT)W^*]B^*\} \\ &= \mathrm{Re}\,\{(X-T)W^* + [B^*(AX-BT)]W^*\} \\ &= \mathrm{Re}\,\{(X-T)W^* + [(B^*A)X - (B^*B)T]W^*\} \\ &= \mathrm{Re}\,\{[X-T+(B^*A)X-(B^*B)T]W^*\}\,. \end{aligned}$$

Therefore, by (4) (iii) in Appendix 1, condition (A.1) implies that

$$X - T + (B^*A)X - (B^*B)T = 0\,,$$

and hence

$$\text{(A.2)} \qquad T = (1+B^*A)X/(1+B^*B)\,.$$

Now, the squared length of the vector (X, AX) is

$$|(X, AX)|^2 = XX^* + (AX)(AX)^* = XX^* + AA^*XX^*,$$

i.e.,

$$|(X, AX)|^2 = (1+A^*A)XX^*. \tag{A.3}$$

Similarly,

$$|(T, BT)|^2 = (1+B^*B)TT^*.$$

But by (A.2),

$$\begin{aligned} TT^* &= (1+B^*A)X[(1+B^*A)X]^*/(1+B^*B)^2 \\ &= (1+B^*A)(1+A^*B)XX^*/(1+B^*B)^2. \end{aligned}$$

Therefore,

$$|(T, BT)|^2 = (1+B^*A)(1+A^*B)XX^*/(1+B^*B). \tag{A.4}$$

Hence, it follows from (A.3) and (A.4) that the angle θ between the vector $(X, AX) \in \mathbf{A}$ and its projection on $\mathbf{B}$ is given by

$$\cos^2\theta = \frac{|(T, BT)|^2}{|(X, AX)|^2} = \frac{(1+A^*B)(1+B^*A)}{(1+A^*A)(1+B^*B)},$$

which shows that the angle between any nonzero vector $(X, AX) \in \mathbf{A}$ and its projection on $\mathbf{B}$ is independent of the choice of X; that is, the 8-plane $\mathbf{A}$ is isoclinic with the 8-plane $\mathbf{B}$.

Finally, to show that the 8-plane $\mathbf{A}$: $Y = AX$ is isoclinic with the 8-plane $\mathbf{O}^\perp$: $X = 0$, we need only observe that the projection of the nonzero vector $(X, AX) \in \mathbf{A}$ on $\mathbf{O}^\perp$ is the vector (O, AX), and

$$\frac{|(O, AX)|^2}{|(X, AX)|^2} = \frac{(AX)(AX)^*}{(1+A^*A)XX^*} = \frac{AA^*}{1 + AA^*}$$

is independent of X.

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(Reçu le 8 septembre 1987)

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