3. The sphère bundles $S^{2n-1} \rightarrow \Phi_n, \quad n=2,4,or 8$, WITH FIBERS ON MUTUALLY ISOCLINIC n-PLANES IN R^{2n}

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.07.2024**

$I = f(0)$ of G, then the velocity vector $f'(0)$ of this curve at I is an element of g. Now

$$t \to f_i(t) \equiv (\cos t)I + (\sin t)B_i \qquad (i=1, ..., 7)$$

are obviously curves in G such that $f_i(0) = I$ and $f_i'(0) = B_i$. Therefore, B_i are all elements of g.

Since g is a Lie subalgebra of $o(8)$ and $B_i \in g$, the Lie products $[B_i, B_j] = B_iB_j - B_jB_i = 2B_iB_j$, where $i, j = 1, ..., 7$, and $i < j$, are all in g.

We have thus proved that the 28 linearly independent skew-symmetric matrices, B_i, B_iB_j all belong to $g \subset o(8)$. Since $o(8)$ is the Lie algebra of all skew-symmetric matrices of order 8 and is therefore of dimension 28, g coincides with $o(8)$. This completes the proof of Theorem 2.6.

3. The sphere bundles $S^{2n-1} \to \Phi_n$, $n = 2, 4$, or 8, with fibers on mutually isoclinic n-planes in R^{2n}

In R^{2n}, $n = 2, 4$, or 8, provided with rectangular coordinate system (x, y), let S^{2n-1} be the unit sphere and Φ_n the maximal set of mutually isoclinic n-planes $\{x = 0, y = xB(\lambda)\}$ defined in Theorem 1.6. Then with the preparations we have made in § 2, we can now prove

THEOREM 3.1. *In* R^{2n}, $n = 2, 4$, *or* 8, *the n-planes in the maximal set* Φ_n *of mutually isoclinic n-planes slice the unit sphere* S^{2n-1} *into a fiber bundle*

$$\mathscr{I}_n = (S^{2n-1}, \Phi_n, \pi, S^{n-1}, G_n),$$

with base space Φ_n, *projection* π, *fiber* S^{n-1} *and group* G_n, *where* $G_2 = S^1$, $G_4 = S^3$, *and* $G_8 = SO(8)$.

Proof. We prove by exhibiting all the ingredients of a representative coordinate bundle.

(1) The bundle space S^{2n-1} has the equation $xx^T + yy^T = 1$ in R^{2n}.

(2) The base space Φ_n is covered by the two coordinate systems

$$(\Phi_n \backslash \mathbf{O}^{\perp}, \lambda), \quad (\Phi_n \backslash \mathbf{O}, \mu) \tag{2.5}$$

as in the proof of Theorem 2.3, where $\mathbf{O}^{\perp}$ is the n-plane $x = 0$, $\mathbf{O}$ is the n-plane $y = 0$, λ is the parameter in the equation $y = xB(\lambda)$ of an n-plane in $\Phi_n \backslash \mathbf{O}^{\perp}$, and μ is the parameter in the equation $x = yB(\mu)^T$ of an n-plane in $\Phi_n \backslash \mathbf{O}$. Moreover, for an n-plane in the intersection $\Phi_n \backslash \{\mathbf{O}^\perp, \mathbf{O}\}$ of the two coordinate neighborhoods, its two coordinates λ and μ, both nonzero, are related by

$$\mu = \lambda/N(\lambda), \quad \text{or equivalently,} \quad \lambda = \mu/N(\mu). \tag{2.6}$$

(3) The projection $\pi: S^{2n-1} \to \Phi_n$ is the map which sends a point of S^{2n-1} to the unique n-plane in Φ_n containing this point (cf. Theorems 1.2 and 1.4).

To see that π is continuous, we let

$$V_1 = \{(x, y) \in S^{2n-1} : x \neq 0\}, \qquad V_2 = \{(x, y) \in S^{2n-1} : y \neq 0\}.$$

Then $\{V_1, V_2\}$ is an open cover of S^{2n-1}, and $\pi(V_1) = \Phi_n \backslash \mathbf{O}^\perp$, $\pi(V_2) = \Phi_n \backslash \mathbf{O}$. Now by Theorem 2.2, the restriction $\pi \mid V_1$ of π to V_1 sends a point $(u, v) \in V_1 \subset S^{2n-1}$ to the n-plane $y = xB(\lambda)$ in $\Phi_n \backslash \mathbf{O}^\perp$ with coordinate

$$\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) = (vu^T, -vB_1u^T, \ldots, -vB_{n-1}u^T)/(uu)^T.$$

This shows that $\pi \mid V_1$ is continuous. Similarly for $\pi \mid V_2$. Therefore, π is continuous.

(4) The fiber S^{n-1} is the unit sphere $tt^T = 1$ in R^n. Here, $t = [t_1 \ldots t_n]$ is a rectangular coordinate system in R^n.

(5) The group G_n of the bundle is $G_2 = S^1 = SO(2)$, $G_4 = S^3 = SO(4)$, or $G_8 = SO(8)$, for $n = 2, 4$, or 8, respectively.

To see that G_n acts on S^{n-1} effectively, we need only observe that if M is an element of $G_n \subset SO(n)$ such that $tM = t$ for all t with $tt^T = 1$, then $M = I$.

(6) With the coordinate systems (2.5) covering the base space Φ_n as described in (2), the coordinate functions are the maps

$$\phi_1 : (\Phi_n \backslash \mathbf{O}^\perp) \times S^{n-1} \to \pi^{-1}(\Phi_n \backslash \mathbf{O}^\perp),$$
$$\phi_2 : (\Phi_n \backslash \mathbf{O}) \times S^{n-1} \to \pi^{-1}(\Phi_n \backslash \mathbf{O}),$$

defined respectively by

$$\phi_1(\lambda, t) = (x, y) = \frac{(t, tB(\lambda))}{\sqrt{1 + N(\lambda)}}, \tag{3.1}$$

$$\phi_2(\mu, t') = (x', y') = \frac{(t'B(\mu)^T, t')}{\sqrt{1 + N(\mu)}}. \tag{3.2}$$

Here, the λ in $\phi_1(\lambda, t)$ denotes the n-plane in $\Phi_n \backslash \mathbf{O}^\perp$ with equation $y = xB(\lambda)$, and the μ in $\phi_2(\mu, t')$ denotes the n-plane in $\Phi_n \backslash \mathbf{O}$ with equation $x = yB(\mu)^T$.

To justify our definition, we must show that ϕ_1, ϕ_2 are homeomorphisms. Obviously, they are continuous maps. To find ϕ_1^{-1} which sends (x, y) to (λ, t), we first note that $x \neq 0$ (cf. (3)) and the last equation (3.1) is equivalent to

$$y = xB(\lambda), \qquad t = x\sqrt{1 + N(\lambda)}\,. \tag{3.3}$$

Now, equation $(3.3)_2$ gives t as a continuous function of x and λ, and by Theorem 2.2, equation $(3.3)_1$ determines λ as a continuous function of x and y. Therefore, λ and t are continuous functions of x and y. This proves that ϕ_1^{-1} is well defined and is continuous, and consequently, ϕ_1 is a homeomorphism. Similarly for ϕ_2.

(7) The projection π and the coordinate functions ϕ_1, ϕ_2 as defined in (3) and (6) satisfy the conditions

$$(\pi \circ \phi_1)(\lambda, t) = \lambda\,, \qquad (\pi \circ \phi_2)(\mu, t') = \mu\,. \tag{3.4}$$

In fact, from (3.1) and (3.3), we see that the point $(x, y) = \phi_1(\lambda, t)$ of S^{2n-1} lies on the n-plane $y = xB(\lambda)$. Therefore, by (3), $\pi(x, y)$ is the n-plane $y = xB(\lambda)$ in $\Phi_n \backslash \mathbf{O}^\perp$ with coordinate λ. This proves $(3.4)_1$. Similarly for $(3.4)_2$.

(8) Let $\mathbf{B}$ be any fixed n-plane in $\Phi_n \backslash \{\mathbf{O}^\perp, \mathbf{O}\}$ with coordinate λ in $\Phi_n \backslash \mathbf{O}^\perp$ and coordinate μ in $\Phi_n \backslash \mathbf{O}$, and let $\phi_{1,\mathbf{B}}$ and $\phi_{2,\mathbf{B}}$ be the two homeomorphisms $S^{n-1} \to \pi^{-1}(\mathbf{B}) \subset S^{2n-1}$ defined by

$$\phi_{1,\mathbf{B}}(t) = \phi_1(\lambda, t)\,, \qquad \phi_{2,\mathbf{B}}(t') = \phi_2(\mu, t')\,.$$

Then $\phi_{2,\mathbf{B}}^{-1} \circ \phi_{1,\mathbf{B}}$ is a homeomorphism in the fiber S^{n-1}, called a coordinate transformation.

We now show that this coordinate transformation coincides with the action of an element of the group G_n. Suppose that t is any point of S^{n-1} and

$$(\phi_{2,\mathbf{B}}^{-1} \circ \phi_{1,\mathbf{B}})(t) = t' \in S^{n-1}\,.$$

Then

$$\phi_{1,\mathbf{B}}(t) = \phi_{2,\mathbf{B}}(t')\,, \qquad \text{i.e.,} \qquad \phi_1(\lambda, t) = \phi_2(\mu, t')\,.$$

Now, by (3.1) and (3.2), this equation is the same as

$$\text{(3.5)} \qquad \frac{(t,\, tB(\lambda))}{\sqrt{1+N(\lambda)}} = \frac{(t'B(\mu)^T,\, t')}{\sqrt{1+N(\mu)}}.$$

Since the two coordinates λ, μ of the n-plane $\mathbf{B}$ satisfy the conditions

$$\lambda \neq 0\,, \quad \mu \neq 0\,, \quad B(\lambda)^{-1} = B(\mu)^T\,,$$
$$\mu = \lambda/N(\lambda)\,, \quad \lambda = \mu/N(\mu)\,, \quad N(\lambda)N(\mu) = 1\,,$$

we can easily verify that equation (3.5) is equivalent to

$$\text{(3.6)} \qquad t' = tB(\lambda)/N(\lambda)^{1/2}\,,$$

and this, on putting $\lambda' = \lambda/N(\lambda)^{1/2}$, we can write as

$$\text{(3.6')} \qquad t' = tB(\lambda')\,, \quad \text{where} \quad N(\lambda') = 1\,.$$

The transformation (3.6), or equivalently, (3.6′), is then a coordinate transformation in the fiber S^{n-1}. Now, by Theorems 2.4, 2.5 and 2.6, G_n is the subgroup of $SO(n)$ generated by the set of elements $\{B(\lambda'): N(\lambda') = 1\}$ of $SO(n)$. Therefore, the coordinate transformation (3.6′) coincides with the action of an element of G_n.

(9) Finally, we see from (3.6) that the map

$$(\Phi_n \backslash \mathbf{O}^\perp) \cap (\Phi_n \backslash \mathbf{O}) = \Phi_n \backslash \{\mathbf{O}^\perp, \mathbf{O}\} \to G_n\,,$$

defined by $\mathbf{B} \to \phi_{2,\mathbf{B}}^{-1} \circ \phi_{1,\mathbf{B}}$, can be expressed in coordinates as

$$\lambda \to B(\lambda)/N(\lambda)^{1/2}\,.$$

Therefore, it is continuous.

Thus, with the ingredients (1)-(9) exhibited above, we have constructed a representative coordinate bundle of the sphere bundle $\mathscr{I}_n$ in Theorem 3.1.

REMARK 1. In Theorem 2.3, we have shown that Φ_n is diffeomorphic with S^n. Therefore, the three sphere bundles $\mathscr{I}_n$ in Theorem 3.1 are topologically the same as some sphere bundles $S^{2n-1} \to S^n$ by S^{n-1}. In fact, we shall prove in § 5 that they are topologically essentially the same as the three Hopf-Steenrod sphere bundles.

REMARK 2. The coordinate functions ϕ_1 and ϕ_2 which we used in (6) were not accidentally come by. They were obtained in the following way. By definition, the coordinate function

$$\phi_1 : (\Phi_n \backslash \mathbf{O}^{\perp}) \times S^{n-1} \to \pi^{-1}(\Phi_n \backslash \mathbf{O}^{\perp}) \subset S^{2n-1}$$

is a homeomorphism sending

$$(\lambda, t) \to (x, y) \in \pi^{-1}(\Phi_n \backslash \mathbf{O}^{\perp}),$$

so that x, y are some continuous functions of λ and t, and λ, t are some continuous functions of x and y. These functions are not arbitrary, but should satisfy certain conditions. First, they must be such that $(\pi \circ \phi_1)(\lambda, t) = \pi(x, y) = \lambda$ (cf. (7)). Therefore, x and y must be related by

$$y = xB(\lambda). \tag{3.7}$$

Secondly, since $(x, y) \in S^{2n-1}$, we must have $xx^T + yy^T = 1$. Furthermore, because of (3.7) and Theorem 2.1 (i),

$$yy^T = xB(\lambda)\left(xB(\lambda)\right)^T = xx^T N(\lambda).$$

Therefore,

$$xx^T = \left(1 + N(\lambda)\right)^{-1}. \tag{3.8}$$

Finally, since $t \in S^{n-1}$, we must have

$$tt^T = 1. \tag{3.9}$$

Conditions (3.7), (3.8) and (3.9) suggest that the simplest possible choice of the continuous functions x, y of λ and t which define our ϕ_1 are those given in (3.1). Similarly for ϕ_2.

With slight modification, we can prove

THEOREM 3.2. *In R^{2n}, $n = 2, 4,$ or 8, the n-planes in the maximal set Φ_n of mutually isoclinic n-planes slice the space $R^{2n} \backslash O$ into a fiber bundle*

$$\mathscr{I}\mathscr{L}_n = (R^{2n} \backslash O, \Phi_n, \pi, R^n \backslash O, G_n \times \rho_n)$$

with base space Φ_n, projection π, fiber $R^n \backslash O$ and group $G_n \times \rho_n$, where $G_2 = S^1$, $G_4 = S^3$ and $G_8 = SO(8)$, and ρ_n is the group of similitudes in $R^n \backslash O$.

Here, by a similitude in $R^n \backslash O$, we mean a transformation of the form $t \to t\rho$, where ρ is a positive real number.

Proof. The proof is similar to that of Theorem 3.1 but with the following difference. The bundle space is $R^{2n} \backslash O$ provided with a rectangular coordinate system (x, y), and the fiber is $R^n \backslash O$ provided with a rectangular coordinate system t; whereas, the base space Φ_n, with coordinate systems (2.5), is the same as that in the bundle $\mathscr{I}_n$. The projection π is the map which sends a point of $R^{2n}\backslash O$ to the (unique) n-plane in Φ_n containing this point, and the two coordinate functions

$$\phi_1 : (\Phi_n\backslash \mathbf{O}^\perp) \times (R^n\backslash O) \to \pi^{-1}(\Phi_n\backslash \mathbf{O}^\perp),$$
$$\phi_2 : (\Phi_n\backslash \mathbf{O}) \times (R^n\backslash O) \to \pi^{-1}(\Phi_n\backslash \mathbf{O})$$

are defined respectively by

$$\phi_1(\lambda, t) = (x, y) = \big(t, tB(\lambda)\big), \tag{3.10}$$

$$\phi_2(\mu, t') = (x', y') = \big(t'B(\mu)^T, t'\big). \tag{3.11}$$

It readily follows from (3.10) and (3.11) that, for any fixed $\mathbf{B} \in \Phi_n\backslash\{\mathbf{O}^\perp, \mathbf{O}\}$ with coordinate λ in $\Phi_n\backslash\mathbf{O}^\perp$, the coordinate transformation $\phi_{2,\mathbf{B}}^{-1} \circ \phi_{1,\mathbf{B}}$ in the fiber $R^n\backslash O$ sends t to

$$t' = tB(\lambda), \tag{3.12}$$

which, because $\lambda \neq 0$, can be written

$$t' = \big(tB(\lambda)/N(\lambda)^{1/2}\big)N(\lambda)^{1/2}. \tag{3.12'}$$

Since $B(\lambda)/N(\lambda)^{1/2} \in G_n$ and $t \to tN(\lambda)^{1/2}$ is a similitude in $R^n\backslash O$, (3.12′) shows that the coordinate transformation $t \to t'$ coincides with the action of an element of $G_n \times \rho_n$. Finally, by (3.12), the map $\Phi_n\backslash\{\mathbf{O}^\perp, \mathbf{O}\} \to G_n \times \rho_n$ defined by $\mathbf{B} \to \phi_{2,\mathbf{B}}^{-1} \circ \phi_{1,\mathbf{B}}$ can be expressed as $\lambda \to B(\lambda)$, and is therefore continuous.

The relationship between the bundle $\mathscr{I}_n$ in Theorem 3.1 and the bundle $\mathscr{I}\mathscr{L}_n$ in Theorem 3.2 is described in the following

THEOREM 3.3.

(i) *The bundle*

$$\mathscr{I}\mathscr{L}_n = (R^{2n}\backslash O, \Phi_n, \pi, R^n\backslash O, G_n \times \rho_n)$$

is equivalent in $G_n \times \rho_n$ *to the bundle*

$$\mathscr{I}\mathscr{L}'_n = (R^{2n}\backslash O, \Phi_n, \pi, R^n\backslash O, G_n)$$

with group G_n.

(ii) *The bundle*

$$\mathscr{I}_n = (S^{2n-1}, \Phi_n, \pi, S^{n-1}, G_n)$$

is a subbundle of the bundle $\mathscr{I}\mathscr{L}'_n$ *in* (i).

Proof. (i) This is an immediate consequence of a result of Steenrod in [5, p. 56, § 12.6]. In fact, from (6) and (8) in the proof of Theorem 3.1, we easily see that the coordinate functions

$$\phi'_1 : (\Phi_n \backslash \mathbf{O}^\perp) \times (R^n \backslash O) \to \pi^{-1}(\Phi_n \backslash \mathbf{O}^\perp),$$
$$\phi'_2 : (\Phi_n \backslash \mathbf{O}) \times (R^n \backslash O) \to \pi^{-1}(\Phi_n \backslash \mathbf{O})$$

of $\mathscr{I}\mathscr{L}'_n$ can be defined respectively by

$$\phi'_1(\lambda, t) = \frac{(t, tB(\lambda))}{\sqrt{1 + N(\lambda)}},$$

$$\phi'_2(\mu, t') = \frac{(t'B(\mu)^T, t')}{\sqrt{1 + N(\mu)}},$$

and that for any fixed element $\mathbf{B} \in \Phi_n \backslash \{\mathbf{O}^\perp, \mathbf{O}\}$, the coordinate transformation $\phi'^{-1}_{2,\mathbf{B}} \circ \phi'_{1,\mathbf{B}}$ in $R^n \backslash O$ is $t \to t' = tB(\lambda)/N(\lambda)^{1/2}$, and thus it cooincides with the action of an element of G_n.

(ii) Obviously, $S^{n-1} \subset R^n \backslash O$ is invariant under G_n. Therefore, according to a result of Steenrod [5, p. 24, 2nd paragraph], there is a unique subbundle of $\mathscr{I}\mathscr{L}'_n$ with fiber S^{n-1} and the same coordinate neighborhoods and coordinate transformations as $\mathscr{I}\mathscr{L}'_n$. Comparison will show that this subbundle is precisely our $\mathscr{I}_n$.

4. A unified treatment of the three Hopf-Steenrod bundles

In the early 30's, H. Hopf [2, 3], using complex numbers, quaternions, and Cayley numbers, discovered his fiberings of S^{2n-1} by S^{n-1} over S^n, $n = 2, 4, 8$. Later in 1950, N. Steenrod [5, pp. 105-110] used these fiberings of Hopf to construct three sphere bundles, which we here call the *Hopf-Steenrod bundles.* But he did this in a roundabout way. For the two cases $n = 2, 4$, he obtained the bundles $S^3 \to S^2$ and $S^7 \to S^4$ as special cases of a general result on "sphere as a bundle over a projective space". For the case $n = 8$, he obtained the bundle $S^{15} \to S^8$ as a subbundle of a linear bundle which he constructed by using Cayley numbers. This being the case, he did not need to define the coordinate functions for his bundles. Still later in 1952, P.J. Hilton [1, pp. 52-55] showed, in a direct manner, that the Hopf fiberings $S^{2n-1} \to S^n$, $n = 2, 4, 8$, are fiber spaces by exhibiting some sets of coordinate functions. But he did not calculate the coordinate