4. A UNIFIED TREATMENT OF THE THREE HOPF-STEENROD BUNDLES

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Proof. (i) This is an immediate consequence of a result of Steenrod in [5, p. 56, § 12.6]. In fact, from (6) and (8) in the proof of Theorem 3.1, we easily see that the coordinate functions

$$\phi'_1 : (\Phi_n \backslash \mathbf{O}^\perp) \times (R^n \backslash O) \to \pi^{-1}(\Phi_n \backslash \mathbf{O}^\perp),$$
$$\phi'_2 : (\Phi_n \backslash \mathbf{O}) \times (R^n \backslash O) \to \pi^{-1}(\Phi_n \backslash \mathbf{O})$$

of $\mathscr{I}\mathscr{L}'_n$ can be defined respectively by

$$\phi'_1(\lambda, t) = \frac{(t, tB(\lambda))}{\sqrt{1 + N(\lambda)}},$$

$$\phi'_2(\mu, t') = \frac{(t'B(\mu)^T, t')}{\sqrt{1 + N(\mu)}},$$

and that for any fixed element $\mathbf{B} \in \Phi_n \backslash \{\mathbf{O}^\perp, \mathbf{O}\}$, the coordinate transformation $\phi'^{-1}_{2,\mathbf{B}} \circ \phi'_{1,\mathbf{B}}$ in $R^n \backslash O$ is $t \to t' = tB(\lambda)/N(\lambda)^{1/2}$, and thus it cooincides with the action of an element of G_n.

(ii) Obviously, $S^{n-1} \subset R^n \backslash O$ is invariant under G_n. Therefore, according to a result of Steenrod [5, p. 24, 2nd paragraph], there is a unique subbundle of $\mathscr{I}\mathscr{L}'_n$ with fiber S^{n-1} and the same coordinate neighborhoods and coordinate transformations as $\mathscr{I}\mathscr{L}'_n$. Comparison will show that this subbundle is precisely our $\mathscr{I}_n$.

4. A unified treatment of the three Hopf-Steenrod bundles

In the early 30's, H. Hopf [2, 3], using complex numbers, quaternions, and Cayley numbers, discovered his fiberings of S^{2n-1} by S^{n-1} over S^n, $n = 2, 4, 8$. Later in 1950, N. Steenrod [5, pp. 105-110] used these fiberings of Hopf to construct three sphere bundles, which we here call the *Hopf-Steenrod bundles.* But he did this in a roundabout way. For the two cases $n = 2, 4$, he obtained the bundles $S^3 \to S^2$ and $S^7 \to S^4$ as special cases of a general result on "sphere as a bundle over a projective space". For the case $n = 8$, he obtained the bundle $S^{15} \to S^8$ as a subbundle of a linear bundle which he constructed by using Cayley numbers. This being the case, he did not need to define the coordinate functions for his bundles. Still later in 1952, P.J. Hilton [1, pp. 52-55] showed, in a direct manner, that the Hopf fiberings $S^{2n-1} \to S^n$, $n = 2, 4, 8$, are fiber spaces by exhibiting some sets of coordinate functions. But he did not calculate the coordinate transformations or mention the bundle groups because they were not needed for his purpose.

In this section, we first describe the fiberings of S^{2n-1} by S^{n-1} over S^n, $n = 2, 4, 8$, as Hopf first discovered them, and then, using Hopf's ideas and method and taking into consideration the work of Steenrod and Hilton, we give a unified and explicit formulation of the structures of the three Hopf-Steenrod bundles $S^{2n-1} \to S^n$. In the next section, we shall show how the Hopf-Steenrod bundles are related to the sphere bundles we constructed in § 3.

Let Q_n, $n = 2, 4, 8$, be respectively the (hypercomplex) systems of complex numbers, quaternions and Cayley numbers. (See Appendix 1 for properties of Cayley numbers.) Suppose that I_a, $a = 0, 1, ..., n-1$, are the base elements in Q_n. Then any element X of Q_n can be uniquely expressed as

$$X = \sum_{a=0}^{n-1} x_{a+1} I_a,$$

where $x_1, ..., x_n$ are real numbers called the components of X. Furthermore, let us define

$$|X| = \left(\sum_{a=0}^{n-1} x_{a+1}^2\right)^{1/2}$$

as the length of X. Then we can identify Q_n with the Euclidean n-space R^n by taking the components $(x_1, ..., x_n)$ of an element X in Q_n as the rectangular coordinates of the point X in R^n.

Consider now the space $Q_n \times Q_n$ of ordered pairs (X, Y) of elements of Q_n, and let $x = (x_1, ..., x_n)$ and $y = (x_{n+1}, ..., x_{2n})$ be the components of X and Y. Then we can identify $Q_n \times Q_n$ with R^{2n} by taking $(x, y) = (x_1, ..., x_n; x_{n+1}, ..., x_{2n})$ as rectangular coordinates in R^{2n}. Calling (X, Y) the Q_n-coordinates in R^{2n}, we define a *Q_n-line* in R^{2n} as either the point set $X = 0$, or a set of all the points whose Q_n-coordinates (X, Y) satisfy an equation of the form $Y = CX$, where C is some element of Q_n. We can easily see that the Q_n-lines are n-planes in R^{2n} with the properties that through any point in $R^{2n}\backslash O$, there passes one and only one such n-plane, and that any two such n-planes intersect only at the origin of R^{2n}.

Suppose that S^{2n-1} is the unit sphere $|X|^2 + |Y|^2 = 1$ in R^{2n}. Then it follows from the above that the great $(n-1)$-spheres in which S^{2n-1} is intersected by the Q_n-lines are such that one and only one of them passes through each point of S^{2n-1}, and so they form a fibering of S^{2n-1} by S^{n-1}.

Closely associated with this fibering of S^{2n-1} is a map p from S^{2n-1} onto the n-sphere S^n, defined as follows. First, we regard S^n as R^n closed by the point ∞ at infinity, so that $S^n = R^n \cup \infty = Q_n \cup \infty$. Then p sends each point of S^{2n-1} which lies on a Q_n-line $Y = CX$ to the point $C \in Q_n \subset S^n$, and sends each point of S^{2n-1} which lies on the Q_n-line $X = 0$ to the point $\infty \in S^n$. In other words, the map $p: S^{2n-1} \to S^n$ is defined by

$$p(X, Y) = \begin{cases} YX^{-1} & \text{if} \quad X \neq 0\,, \\ \infty & \text{if} \quad X = 0\,, \end{cases} \tag{4.1}$$

where (X, Y) is any point of S^{2n-1}. It is easy to see that p is a continuous map, and that the inverse image of each point of S^n is one of the great $(n-1)$-spheres in which S^{2n-1} is intersected by the Q_n-lines.

The fibering $S^{2n-1} \to S^n$ by S^{n-1} constructed above is then the famous *Hopf fibering*, and the map p is the *Hopf map* related to it.

We now prove the following theorem which gives a unified and explicit formulation of the three Hopf-Steenrod bundles $S^{2n-1} \to S^n$, $n = 2, 4, 8$.

THEOREM 4.1. *Let* Q_n, $n = 2, 4, 8$, *be respectively the systems of complex numbers, quaternions and Cayley numbers, and let the spaces* Q_n, $Q_n \times Q_n$ *be identified with* R^n, $R^n \times R^n = R^{2n}$, *respectively. Then the set of* Q_n*-lines* $\{X = 0, Y = CX\}$ *in* R^{2n} *slice the unit sphere* S^{2n-1} *in* R^{2n} *into a fiber bundle*

$$\mathscr{H}\mathscr{S}_n = (S^{2n-1}, S^n, p, S^{n-1}, O(n))$$

with base space $S^n = Q_n \cup \infty$, *projection* p, *fiber* S^{n-1} *and group the orthogonal group* $O(n)$.

Proof. We prove by exhibiting the ingredients of a representative coordinate bundle.

(1) The bundle space S^{2n-1} is the unit sphere $|X|^2 + |Y|^2 = 1$ in $R^{2n} = Q_n \times Q_n$.

(2) The base space S^n is identified with $R^n \cup \infty = Q_n \cup \infty$. Therefore, S^n is covered by the two coordinate neighborhoods

$$Q_n\,, \qquad S^n \backslash O = (Q_n \cup \infty) \backslash O = (Q_n \backslash O) \cup \infty\,,$$

with elements of Q_n and ∞ serving as coordinates.

(3) The projection $p: S^{2n-1} \to S^n$ is the Hopf map defined by (4.1).

(4) The fiber S^{n-1} is the unit sphere $|X| = 1$ in $R^n = Q_n$.

(5) The group $O(n)$ of the bundle acts on S^{n-1} effectively.

(6) Let C, D be elements of Q_n such that $|D| = 1$, so that D represents a point of S^{n-1}. Then the two coordinate functions are the homeomorphisms

$$\psi_1 : Q_n \times S^{n-1} \to p^{-1}(Q_n),$$
$$\psi_2 : (S^n \backslash O) \times S^{n-1} \to p^{-1}(S^n \backslash O),$$

defined respectively by

$$\psi_1(C, D) = \frac{(D, CD)}{\sqrt{1 + |C|^2}}, \tag{4.2}$$

$$\begin{cases} \psi_2(C, D) = \dfrac{(C^{-1}D, D)}{\sqrt{1 + 1/|C|^2}}, \quad \text{where} \quad C \neq \infty, \\ \\ \psi_2(\infty, D) = (O, D). \end{cases} \tag{4.3}$$

That ψ_1, ψ_2 are indeed homeomorphisms is easy to verify.

(7) It can readily be seen from (4.1), (4.2) and (4.3) that the projection p and the coordinate functions ψ_1, ψ_2 satisfy the conditions:

$$\begin{cases} (p \circ \psi_1)(C, D) = C, \\ (p \circ \psi_2)(C, D) = C \quad \text{if} \quad C \neq \infty, \quad \text{and} \quad (p \circ \psi_2)(\infty, D) = \infty. \end{cases} \tag{4.4}$$

(8) For each fixed point C in the intersection $Q_n \cap (S^n \backslash O) = Q_n \backslash O$ of the two coordinate neighborhoods in the base space S^n, let $\psi_{1,C}$ and $\psi_{2,C}$ be the two homeomorphisms $S^{n-1} \to p^{-1}(C) \subset S^{2n-1}$ defined by

$$\psi_{1,C}(D) = \psi_1(C, D), \quad \psi_{2,C}(D) = \psi_2(C, D),$$

where ψ_1, ψ_2 are the coordinate functions defined in (6). Then, we can easily verify by using (4.2) and (4.3) that the coordinate transformation $\psi_{2,C}^{-1} \circ \psi_{1,C}$ in the fiber S^{n-1} is

$$D \to CD/|C|, \tag{4.5}$$

where D with $|D| = 1$ is a variable point of $S^{n-1} \subset Q_n$. Now since the components of the product CX of any two elements C, X of Q_n are bilinear functions of the components of C, X, the map $X \to CX/|C|$ is a linear transformation in $R^n = Q_n$. It is in fact an orthogonal transformation because $|CX/|C|| = |X|$. Therefore, the coordinate transformation (4.5) coincides with the action of an element of the group $O(n)$.

(9) Finally, from the bilinearity of the product CX, it also follows that the coordinate transformation (4.5) varies continuously with C. Therefore, the

map from $Q_n \cap (S^n \backslash O) = Q_n \backslash O \to O(n)$ defined by $C \to \psi_{2,C}^{-1} \circ \psi_{1,C}$ is continuous.

Thus, with the ingredients (1)-(9) exhibited above, we have constructed a representative coordinate bundle of the bundle $\mathscr{H}\mathscr{S}_n$ in the theorem.

REMARK. The coordinate functions ψ_1 and ψ_2 as given in (4.2) and (4.3) were arrived at as follows. By definition, ψ_1 is a homeomorphism sending

$$(C, D) \in Q_n \times S^{n-1} \to (X, Y) \in p^{-1}(Q_n) \subset S^{2n-1}.$$

Here, X and Y are not arbitrary functions of C, D, but must satisfy certain conditions. First, they must satisfy $(4.4)_1$, so that $(p \circ \psi_1)(C, D) = p(X, Y) = C$. Therefore, by (4.1) X and Y are related by

$$Y = CX. \tag{4.6}$$

Secondly, since (X, Y) is a point of S^{2n-1}, $|X|^2 + |Y|^2 = 1$. Combining this with (4.6), we get

$$|X|^2 = 1/(1+|C|^2). \tag{4.7}$$

Finally, since $D \in S^{n-1} \subset Q_n$,

$$|D| = 1. \tag{4.8}$$

Conditions (4.6), (4.7) and (4.8) suggest that the simplest choice of ψ_1 is (4.2). Similarly, we choose (4.3) as ψ_2 because of conditions $(4.4)_2$.

Similar to Theorem 4.1, we have

THEOREM 4.2. *In R^{2n}, $n = 2, 4$, or 8, the Q_n-lines slice $R^{2n} \backslash O$ into a fiber bundle*

$$\mathscr{S}\mathscr{L}_n = (R^{2n} \backslash O, S^n, p, Q_n \backslash O, GL(n, R))$$

with base space $S^n = Q_n \cup \infty$, projection p, fiber $Q_n \backslash O$, and group the general linear group $GL(n, R)$.

Proof. The proof is similar to that of Theorem 4.1, but with the following difference. The projection is the map $p: R^{2n} \backslash O \to S^n$ defined by

$$p(X, Y) = \begin{cases} YX^{-1} & \text{if} \quad X \neq 0, \\ \infty & \text{if} \quad X = 0; \end{cases} \tag{4.9}$$

and the two coordination functions

$$\psi_1 : Q_n \times (Q_n \backslash O) \to p^{-1}(Q_n),$$
$$\psi_2 : (S^n \backslash O) \times (Q_n \backslash O) \to p^{-1}(S^n \backslash O)$$

are defined respectively by

(4.10) $$\psi_1(C, D) = (D, CD),$$

(4.11) $$\begin{cases} \psi_2(C, D) = (C^{-1}D, D), & \text{where} \quad C \neq \infty, \\ \psi_2(\infty, D) = (O, D). \end{cases}$$

The coordinate transformations $\psi_{2,C}^{-1} \circ \psi_{1,C}$, where $C \in Q_n \cap (S^n \backslash O) = Q_n \backslash O$, are the linear maps $D \to CD$ in the fiber $Q_n \backslash O$.

The relationship between the bundles $\mathscr{H}\mathscr{S}_n$ and $\mathscr{S}\mathscr{L}_n$ is described in the following theorem, the proof of which is similar to that of Theorem 3.3.

THEOREM 4.3.

(i) *The bundle*

$$\mathscr{S}\mathscr{L}_n = \left(R^{2n} \backslash O, S^n, p, Q_n \backslash O, GL(n, R)\right)$$

is equivalent in $GL(n, R)$ *to the bundle*

$$\mathscr{S}\mathscr{L}'_n = \left(R^{2n} \backslash O, S^n, p, Q_n \backslash O, O(n)\right)$$

with group $O(n)$.

(ii) *The bundle*

$$\mathscr{H}\mathscr{S}_n = \left(S^{2n-1}, S^n, p, S^{n-1}, O(n)\right)$$

is a subbundle of the bundle $\mathscr{S}\mathscr{L}'_n$.

Let us now explain how the bundle $\mathscr{H}\mathscr{S}_n$ given in Theorem 4.1 is a unified formulation of the sphere bundles $S^{2n-1} \to S^n$, $n = 2, 4, 8$ constructed by N. Steenrod using the Hopf fiberings, and how our construction incorporates the work of P. J. Hilton.

(a) Comparison of the ingredients of the sphere bundle $\mathscr{H}\mathscr{S}_8$ in Theorem 4.1 with those of the fiber space $S^{15} \to S^8$ of Hilton [1, p. 54] shows that they have the same projection (4.1) and coordinate functions (4.2) and (4.3).

(b) Suppose that in the construction of the sphere bundle $\mathscr{H}\mathscr{S}_n$ in Theorem 4.1, we use the "Q_n-lines" $X = CY$ instead of the Q_n-lines $Y = CX$ in defining the projection $p : S^{2n-1} \to S^n$. Then we can obtain another sphere bundle $S^{2n-1} \to S^n$ by using the ingredients of $\mathscr{H}\mathscr{S}_n$ but interchanging the roles of X and Y, i.e., by replacing

(i) the projection (4.1) by

$$p(X, Y) = \begin{cases} XY^{-1} & \text{if} \quad Y \neq 0\,, \\ \infty & \text{if} \quad Y = 0\,, \end{cases} \tag{4.1'}$$

and

(ii) the coordinate functions (4.2) and (4.3) by

$$\psi_1(C, D) = \frac{(CD, D)}{\sqrt{1 + |C|^2}}\,, \tag{4.2'}$$

$$\begin{cases} \psi_2(C, D) = \dfrac{(D, C^{-1}D)}{\sqrt{1 + 1/|C|^2}}\,, & \text{where} \quad C \neq \infty\,, \\ \psi_2(\infty, D) = (D, O)\,. \end{cases} \tag{4.3'}$$

For $n = 2$, the X, Y, C and D (with $|D|=1$) are all complex numbers. On putting $X = z_1$, $Y = z_2$, $C = \mu$ and $D = e^{i\theta}$, we can see immediately that the projection (4.1′) and the coordinate functions (4.2′) and (4.3′) are exactly those used by Hilton [1, p. 51] to prove that the Hopf fibering $S^3 \to S^2$ has a fiber space structure.

(c) Suppose that in the construction of the linear bundle $\mathscr{S}\mathscr{L}_n$ in Theorem 4.2, we use the "Q_n-lines" $X = CY$ instead of the Q_n-lines $Y = CX$ in defining the projection $p\colon R^{2n}\backslash O \to S^n$. Then we can obtain another linear bundle by using the ingredients of $\mathscr{S}\mathscr{L}_n$, but interchanging the roles of X and Y, i.e., by replacing

(i) the projection (4.9) by

$$p(X, Y) = \begin{cases} XY^{-1} & \text{if} \quad Y \neq 0\,, \\ \infty & \text{if} \quad Y = 0\,, \end{cases} \tag{4.9'}$$

and

(ii) the coordinate functions (4.10) and (4.11) by

$$\psi_1(C, D) = (CD, D)\,, \tag{4.10'}$$

$$\begin{cases} \psi_2(C, D) = (D, C^{-1}D)\,, & \text{where} \quad C \neq \infty\,, \\ \psi_2(\infty, D) = (D, O)\,. \end{cases} \tag{4.11'}$$

For $n = 8$, the X, Y, C and D are Cayley numbers. On putting $X = c$, $Y = d$, $C = x$ and $D = y$, we can see immediately that the projection (4.9′) and the coordinate functions (4.10′) and (4.11′) are exactly those of the linear bundle $\mathscr{B}$ constructed by N. Steenrod in [5, pp. 109-110]. Therefore, this linear bundle $\mathscr{B}$ of Steenrod and the linear bundle $\mathscr{S}\mathscr{L}_8$ in Theorem 4.2 are two slightly different representations of the same bundle.

5. Comparison of our bundles with the Hopf-Steenrod bundles

In § 3, we constructed the sphere bundles

$$\mathscr{I}_n = (S^{2n-1}, \Phi_n, \pi, S^{n-1}, G_n), \qquad n = 2, 4, 8,$$

with fibers lying on mutually isoclinic n-planes in R^{2n}. In § 4, we gave a unified treatment of the classical Hopf-Steenrod sphere bundles

$$\mathscr{H}\mathscr{S}_n = \left(S^{2n-1}, S^n, p, S^{n-1}, O(n)\right), \qquad n = 2, 4, 8,$$

using, as N. Steenrod did, the Hopf map and the hypercomplex systems of complex numbers, quaternions and Cayley numbers. In this section we shall prove that (i) the Hopf fibering $S^{2n-1} \to S^n$ and our maximal set of mutually isoclinic n-planes in R^{2n} are equivalent concepts (Theorems 5.1 and 5.2), and (ii) the representative coordinate bundles constructed in § 3 and § 4 for the bundles $\mathscr{I}_n$ and $\mathscr{H}\mathscr{S}_n$ are topologically essentially the same (Theorem 5.3). For convenience, the theorems will be stated and proofs given for the case $n = 8$ only. Similar theorems hold for the cases $n = 2, 4$, and their proofs follow the same line and are simpler.

Theorem 5.1. *For $n = 8$, let us identify the space Q_8 of Cayley numbers with R^8 by regarding the Cayley number*

$$X \equiv (x_1 + x_2 i + x_3 j + x_4 k\,,\ x_5 + x_6 i + x_7 j + x_8 k)$$

as the point in R^8 with rectangular coordinates $(x_1, ..., x_8)$, and the space $Q_8 \times Q_8$ of ordered pairs of Cayley numbers with $R^8 \times R^8 = R^{16}$ by regarding the ordered pair

$$(X, Y) \equiv \big((x_1 + x_2 i + x_3 j + x_4 k\,,\ x_5 + x_6 i + x_7 j + x_8 k)\,,$$
$$(x_9 + x_{10} i + x_{11} j + x_{12} k\,,\ x_{13} + x_{14} i + x_{15} j + x_{16} k)\big)$$

as the point in R^{16} with rectangular coordinates $(x_1, ..., x_8; x_9, ..., x_{16})$. Then, written in terms of $x = [x_1 \dots x_8]$ and $y = [x_9 \dots x_{16}]$,