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Artikel: AN ALGEBRAIC PROOF OF VAN DER WAERDEN'S THEOREM

Kapitel: 1. Semigroup properties of N

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In general this result does not hold for infinite semigroups. We invite the reader to find a counterexample for $S = (\mathbf{Z}, +)$. However it does hold for compact semigroups as we will show implicitly in the proof of Theorem 3.3. (Of course the finite version is then a special case.)

We intend to apply this theorem to the natural numbers $\mathbf{N}$ by compactifying $\mathbf{N}$ in such a way so as to obtain a compact semigroup; this is the role of the Stone-Čech compactification $\beta\mathbf{N}$ of $\mathbf{N}$. We obtain a theorem about $\beta\mathbf{N}$ which when unraveled becomes exactly van der Waerden's Theorem.

We warn the reader that in the compactification of $\mathbf{N}$ the operation of addition will be extended with the usual notation $+$. However the semigroup will not be commutative and so one has to accustom oneself to non-commutative addition.

1. Semigroup properties of $\beta\mathbf{N}$

Any completely regular Hausdorff space has a maximal compactification, the Stone-Čech compactification. In particular the discrete space $\mathbf{N}$ of positive integers has a Stone-Čech compactification $\beta\mathbf{N}$ which is characterized by: (1) $\beta\mathbf{N}$ is a compact Hausdorff space; (2) $\mathbf{N}$ is a dense subset of $\beta\mathbf{N}$; and (3) given any compact Hausdorff space Y and any $f: \mathbf{N} \to Y$ there is a continuous extension $f^{\beta}: \beta\mathbf{N} \to Y$, (that is $f^{\beta}|_{\mathbf{N}} = f$).

Our proof of van der Waerden's Theorem is based on the fact that the operation of ordinary addition extends to $\beta\mathbf{N}$ as an operation which we denote by $+$. $\beta\mathbf{N}$ under this operation will be a semigroup in which the operation of addition is continuous in a restricted way. Namely let $(S, +)$ be a semigroup with S a topological space and define functions ρ_x and λ_x for each $x \in S$ by $\rho_x(y) = y + x$ and $\lambda_x(y) = x + y$. If one requires only that ρ_x be continuous, S is called a right topological semigroup.

1.1 Lemma. *There is an operation* $+$ *on* $\beta\mathbf{N}$ *such that* $\beta\mathbf{N}$ *is a compact right topological semigroup,* $+$ *extends ordinary addition on* $\mathbf{N}$, *and* λ_n *is continuous for each* $n \in \mathbf{N}$.

Proof. We extend $+$ in stages, starting with $+$ defined on $\mathbf{N} \times \mathbf{N}$. Given $n \in \mathbf{N}$, consider $f_n: \mathbf{N} \to \beta\mathbf{N}$ defined by $f_n(m) = n + m$. Then each f_n has a continuous extension $f_n^{\beta}: \beta\mathbf{N} \to \beta\mathbf{N}$. For $n \in \mathbf{N}$ and $p \in \beta\mathbf{N}\backslash\mathbf{N}$ define

$n + p = f_n^\beta(p)$. (Then for $n \in \mathbf{N}$ and any $p \in \beta\mathbf{N}$, $n + p = f_n^\beta(p)$ since if $p \in \mathbf{N}$, $f_n^\beta(p) = f_n(p) = n + p$.) Now $+$ is defined on $\mathbf{N} \times \beta\mathbf{N}$. Given $p \in \beta\mathbf{N}$ define $g_p : \mathbf{N} \to \beta\mathbf{N}$ by $g_p(n) = n + p$. Then each g_p has a continuous extension $g_p^\beta : \beta\mathbf{N} \to \beta\mathbf{N}$. Then for $p \in \beta\mathbf{N}$ and $q \in \beta\mathbf{N}\backslash\mathbf{N}$ define $q + p = g_p^\beta(q)$. (Again if p, q are any points in $\beta\mathbf{N}$ we have $q + p = g_p^\beta(q)$.)

Since for any $n \in \mathbf{N}$, $\lambda_n = f_n^\beta$ and for any $p \in \beta\mathbf{N}$, $\rho_p = g_p^\beta$, the continuity assumptions are immediate. Thus we need only check that the operation is associative. To this end let $p, q, r \in \beta\mathbf{N}$. Observe that $p + (q+r) = \rho_{q+r}(p)$ while $(p+q) + r = (\rho_r \circ \rho_q)(p)$ so by continuity it suffices to show ρ_{q+r} and $\rho_r \circ \rho_q$ agree on the dense subset $\mathbf{N}$ of $\beta\mathbf{N}$. Let $n \in \mathbf{N}$. Then

$$\rho_{q+r}(n) = n + (q+r) = (\lambda_n \circ \rho_r)(q)$$

$$\text{and} \quad (\rho_r \circ \rho_q)(n) = (n+q) + r = (\rho_r \circ \lambda_n)(q).$$

Again by continuity, it suffices to show $\lambda_n \circ \rho_r$ and $\rho_r \circ \lambda_n$ agree on $\mathbf{N}$. Let $m \in \mathbf{N}$. Then

$$(\lambda_n \circ \rho_r)(m) = n + (m+r) = (\lambda_n \circ \lambda_m)(r)$$

while

$$(\rho_r \circ \lambda_n)(m) = (n+m) + r = \lambda_{n+m}(r).$$

Thus it finally suffices to show $\lambda_n \circ \lambda_m$ and λ_{n+m} agree on $\mathbf{N}$. Let $t \in \mathbf{N}$. Then $(\lambda_n \circ \lambda_m)(t) = n + (m+t) = (n+m) + t = \lambda_{n+m}(t)$ as required. □

The main fact about $\beta\mathbf{N}$ making it useful for van der Waerden's Theorem and similar results is the content of the following lemma.

1.2 Lemma. *If* $\{A_1, A_2, ..., A_m\}$ *is a finite partition of* $\mathbf{N}$, *then* $\{cl\, A_1, cl\, A_2, ..., cl\, A_m\}$ *is a partition of* $\beta\mathbf{N}$ *such that for each* $i \in \{1, 2, ..., m\}$, $cl\, A_i$ *is open.*

Proof. Let $Y = \{1, 2, ..., m\}$ with the discrete topology and define $f : \mathbf{N} \to Y$ by $f(n) = i$ if and only if $n \in A_i$. For each $i \in \{1, 2, ..., m\}$, let $B_i = \{p \in \beta\mathbf{N} : f^\beta(p) = i\}$. Then immediately $\{B_1, B_2, ..., B_m\}$ is a partition of $\beta\mathbf{N}$. Further, given $i \in \{1, 2, ..., m\}$, $B_i = (f^\beta)^{-1}[\{i\}]$. Since $\{i\}$ is open and closed in Y and f^β is continuous, B_i is open and closed. Since $A_i \subseteq B_i$, one has $cl\, A_i \subseteq B_i$. To see that $B_i \subseteq cl\, A_i$, let $x \in B_i$ and let U be a neighborhood of x. Since $\mathbf{X}$ is dense in $\beta\mathbf{N}$, pick $y \in \mathbf{N} \cap (U \cap B_i)$. Since $y \in B_i$, $f(y) = i$ so $y \in A_i$. Thus $U \cap A_i \neq \emptyset$ as required. □