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Artikel: AN ALGEBRAIC PROOF OF VAN DER WAERDEN'S THEOREM

Kapitel: 3. Van der Waerden's Theorem

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To see that A is in fact a minimal left ideal, assume we have a left ideal $B \subseteq A$ and pick $x \in B$. Then as above $S + x \in \mathscr{A}$ while $S + x \subseteq S + B \subseteq B \subseteq A$ so $S + x = A$ so $B = A$ $\square$

2.3 *Definition.* Let S be a semigroup. Then $M(S) = \cup \{L\colon L$ is a minimal left ideal of $S\}$.

It is a fact (which we will not need) that if S is a compact Hausdorff right topological semigroup, then $M(S)$ is a two-sided ideal of S.

2.4 LEMMA. *Let S be a compact Hausdorff right topological semigroup and let I be a two-sided ideal of S. Then $M(S) \neq \emptyset$ and $M(S) \subseteq I$.*

Proof. Since S is a left ideal of S it contains by Lemma 2.2 a minimal left ideal so $M(S) \neq \emptyset$. So see that $M(S) \subseteq I$, let $x \in M(S)$. There is a minimal left ideal L of S with $x \in L$. Also choose some $y \in I$. Then $y + x \in L \cap I$ (since I is a right ideal) so $L \cap I \neq \emptyset$. Thus $L \cap I$ is a left ideal contained in L so that $L \cap I = L$. $\square$

The proof of the following lemma is an easy exercise and we omit it.

2.5 LEMMA. *Let S_1 and S_2 be compact right topological semigroups and let $S_1 \times S_2$ have the product topology and coordinatewise operations. Then $S_1 \times S_2$ is a compact right topological semigroup. Given $x \in S_1$ and $y \in S_2$, λ_x and λ_y may or may not be continuous (where $\lambda_x(t) = x + t$). If $\lambda_x\colon S_1 \to S_1$ and $\lambda_y\colon S_2 \to S_2$ are continuous, then $\lambda_{(x,y)}\colon S_1 \times S_2 \to S_1 \times S_2$ is continuous.*

3. VAN DER WAERDEN'S THEOREM

We let $l \in \mathbf{N}$ be fixed throughout and show that given any finite partition of $\mathbf{N}$ some one cell contains a length l arithmetic progression.

3.1 *Definition.* (a) Let $Y = (\beta\mathbf{N})^l$ with the product topology and coordinatewise operations.

(b) $E^* = \{(a, a + d, a + 2d, \ldots, a + (l-1)d)\colon a \in \mathbf{N}$ and $d \in \mathbf{N} \cup \{0\}\}$.

(c) $I^* = \{(a, a + d, a + 2d, \ldots, a + (l-1)d)\colon a, d \in \mathbf{N}\}$.

(d) $E = cl_Y\, E^*$

(e) $I = cl_Y\, I^*$.

Note that by Lemmas 1.1 and 2.5, Y is a compact Hausdorff right topological semigroup and whenever $\mathbf{x} = (x_1, x_2, \ldots, x_l) \in \mathbf{N}^l$, $\lambda_{\mathbf{x}}$ is continuous.

3.2 LEMMA. *E is a compact Hausdorff right topological semigroup and I is a two sided ideal of E.*

Proof. Compactness is immediate and the Hausdorff property and right continuity are inherited from Y. We let $\mathbf{p} = (p_1, p_2, \ldots p_l)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_l)$ be members of E and show that $\mathbf{p} + \mathbf{q} \in E$. We show further that if either $\mathbf{p}$ or $\mathbf{q}$ is in I, then $\mathbf{p} + \mathbf{q} \in I$.

To see that $\mathbf{p} + \mathbf{q} \in E$, let U be a neighborhood of $\mathbf{p} + \mathbf{q}$. By the continuity of $\rho_{\mathbf{q}}$, pick a neighborhood V of $\mathbf{p}$ with $V + \mathbf{q} = \rho_{\mathbf{q}}[V] \subseteq U$. Since $\mathbf{p} \in cl\, E^*$ we may pick $a \in \mathbf{N}$ and $d \in \mathbf{N} \cup \{0\}$ with

$$(a, a+d, a+2d, \ldots, a+(l-1)d) \in V.$$

If $\mathbf{p} \in I$ we may presume $d \neq 0$. Let $\mathbf{x} = (a, a+d, a+2d, \ldots, a+(l-1)d)$. Then $\mathbf{x} \in V$ so $\mathbf{x} + \mathbf{q} \in U$. By the continuity of $\lambda_{\mathbf{x}}$, pick a neighborhood W of $\mathbf{q}$ with $\mathbf{x} + W = \lambda_{\mathbf{x}}[W] \subseteq U$. Since $\mathbf{q} \in cl\, E^*$, pick $b \in \mathbf{N}$ and $c \in \mathbf{N} \cup \{0\}$ (with $c \neq 0$ if $\mathbf{q} \in I$) such that $(b, b+c, b+2c, \ldots, b+(l-1)c) \in W$. Let $\mathbf{y} = (b, b+c, b+2c, \ldots, b+(l-1)c)$. Then $\mathbf{x} + \mathbf{y} \in U \cap E^*$. If either $d \neq 0$ or $c \neq 0$, then $c + d \neq 0$ so $\mathbf{x} + \mathbf{y} \in U \cap I^*$. □

3.3 THEOREM. *Let $p \in M(\beta\mathbf{N})$ and let $\mathbf{p} = (p, p, \ldots, p)$. Then $\mathbf{p} \in I$.*

Proof. We first show that $\mathbf{p} \in E$. Let $U_1 \times U_2 \times \ldots \times U_l$ be a basic neighborhood of $\mathbf{p}$. Then $U_1 \cap U_2 \cap \ldots \cap U_l$ is a neighborhood of p in $\beta\mathbf{N}$. Since $\mathbf{N}$ is dense, pick $a \in \mathbf{N} \cap (U_1 \cap U_2 \cap \ldots \cap U_l)$. Then $(a, a, \ldots, a) \in E^* \cap (U_1 \times U_2 \times \ldots \times U_l)$. Thus $\mathbf{p} \in cl\, E^* = E$.

Since $p \in M(\beta\mathbf{N})$, there is a minimal left ideal L of $\beta\mathbf{N}$ with $p \in L$. Since $E + \mathbf{p}$ is a left ideal of E, pick by Lemma 2.2 a minimal left ideal L^* of E with $L^* \subset E + \mathbf{p}$. Since L^* is closed, hence compact, pick by Lemma 2.1 an idempotent $\mathbf{q} = (q_1, q_2, \ldots, q_l)$ in L^*. Now $\mathbf{q} \in L^* \subseteq E + \mathbf{p}$ so pick some $\mathbf{s} = (s_1, s_2, \ldots, s_l)$ in E with $\mathbf{q} = \mathbf{s} + \mathbf{p}$.

We show that $\mathbf{p} + \mathbf{q} = \mathbf{p}$. To this end let $i \in \{1, 2, \ldots, l\}$. Now $q_i = s_i + p \in L$ so $\beta\mathbf{N} + q_i \subseteq \beta\mathbf{N} + L \subseteq L$. Thus $\beta\mathbf{N} + q_i$ is a left ideal contained in the minimal left ideal L so that $\beta\mathbf{N} + q_i = L$. Thus since $p \in L$ there exists $t_i \in \beta\mathbf{N}$ with $t_i + q_i = p$. But then $p + q_i = t_i + q_i + q_i = t_i + q_i = p$ as required.

Since $\mathbf{p} \in E$ and $\mathbf{q} \in L^*$, a left ideal of E, we have $\mathbf{p} = \mathbf{p} + \mathbf{q} \in L^*$ so that $\mathbf{p} \in M(E)$. Thus by Lemma 2.4, $\mathbf{p} \in I$.

3.4 COROLLARY (van der Waerden). *Let $m \in \mathbf{N}$ and let $\{A_1, A_2, \ldots, A_m\}$ be a partition of $\mathbf{N}$. There exist $i \in \{1, 2, \ldots, m\}$ and $a, d \in \mathbf{N}$ with $\{a, a+d, a+2d, \ldots, a+(l-1)d\} \subseteq A_i$.*

Proof. By Lemma 2.4 $M(\beta\mathbf{N}) \neq \emptyset$ so pick $p \in M(\beta\mathbf{N})$ and let $\mathbf{p} = (p, p, \ldots, p)$. By Lemma 1.2 pick $i \in \{1, 2, \ldots, m\}$ such that $cl\, A_i$ is a neighborhood of p and let $U = cl\, A_i$. Then $U \times U \times \ldots \times U$ is a neighborhood of $\mathbf{p}$ while, by Theorem 2.3, $\mathbf{p} \in I = cl\, I^*$. Pick $a, d \in \mathbf{N}$ with $(a, a+d, a+2d, \ldots, a+(l-1)d) \in U \times U \times \ldots \times U$. Then

$$\{a, a + d, a + 2d, \ldots, a + (l-1)d\} \subseteq U \cap \mathbf{N} = (cl\, A_i) \cap N = A_i . \quad \square$$

We remark that if one starts with the free semigroup on l letters in place of $\mathbf{N}$, essentially the same proof yields the Hales-Jewett Theorem. See [3] for the details.

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