

# Application to the local isometric embedding of a Riemannian manifold

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APPLICATION TO THE LOCAL ISOMETRIC EMBEDDING  
OF A RIEMANNIAN MANIFOLD

(following Hörmander [3], Section 2).

Let  $M$  be a compact  $C^\infty$  manifold of dimension  $n$  and  $g$  a smooth Riemannian metric on  $M$ . In local coordinates, we are thus given a positive definite quadratic form

$$g = \sum_{j,k} g_{jk} dx_j dx_k.$$

The celebrated theorem of Nash [7], which is at the origin of the method, states that for some (large) integer  $N$ , there is an isometric embedding  $u: M \rightarrow \mathbf{R}^N$ , that is an injective map satisfying the system of equations

$$(13) \quad \langle \partial_j u, \partial_k u \rangle = g_{jk} \quad 1 \leq j, k \leq n$$

where  $\partial_j$  stands for  $\partial/\partial x_j$  and  $\langle \cdot, \cdot \rangle$  for the Euclidean scalar product in  $\mathbf{R}^N$ ; thus, any compact Riemannian manifold can be thought as a submanifold of a Euclidean space.

In the proof of this Nash theorem, one first establishes that the set of metrics  $g$  such that the problem can be solved is a dense convex cone in the set of all  $C^\infty$  metrics on  $M$ , and this leads to the following reduced problem (see Hörmander [3] Section 2): show that the equation (13) can be solved for every metric in some neighborhood of a fixed metric  $g^0$ .

To illustrate the method described above, let us show how one can use our theorem to prove this last property locally (and this will give a local isometric embedding  $u: M \rightarrow \mathbf{R}^N$ ).

Let  $\Omega = \{x \in \mathbf{R}^n; |x| < 1\}$  and choose, near some point  $x_0 \in M$ , local coordinates such that  $\Omega$  describes a neighborhood of  $x_0$ ; we take a  $C_0^\infty u_0: \mathbf{R}^n \rightarrow \mathbf{R}^{n(n+3)/2}$  equal to

$$((x_j)_{1 \leq j \leq n}, (x_j^2/2)_{1 \leq j \leq n}, (x_j x_k)_{1 \leq j < k \leq n})$$

in a neighborhood of  $\bar{\Omega}$ ; this  $u_0$  is an isometric embedding for the corresponding metric  $g^0$  in  $\Omega$ , namely the metric  $g_{jj}^0 = 1 + |x|^2$  and  $g_{jk} = x_j x_k$  if  $j \neq k$ . Finally, for a metric  $g$  close to  $g^0$ , we consider the restriction  $\phi(u)$  to  $\Omega$  of the function

$$(14) \quad (\langle \partial_j u, \partial_k u \rangle - g_{jk})_{1 \leq j \leq k \leq n}$$

which is a function in  $H^\infty(\Omega)$  valued in  $\mathbf{R}^{n(n+1)/2}$  for any  $u \in H^\infty(\mathbf{R}^n)$  valued in  $\mathbf{R}^{n(n+3)/2}$ . Classically, estimates such as (1) hold for  $s > (n+2)/2$ .

The derivative of  $\phi$  with respect to  $u$  is defined by

$$(15) \quad \phi'(u)v = (\langle \partial_j u, \partial_k v \rangle + \langle \partial_k u, \partial_j v \rangle)_{1 \leq j \leq k \leq n}.$$

If  $\phi \in H^\infty(\Omega)$  is valued in  $\mathbf{R}^{n(n+1)/2}$ , let us consider it as a function valued in  $\mathbf{R}^{n(n+3)/2}$  by adding  $n$  components  $\phi_j = 0$  for  $1 \leq j \leq n$ , and define  $\psi(u)\phi$  as a continuous extension to  $\mathbf{R}^n$  of the function

$$(16) \quad v = -\frac{1}{2} A(u)^{-1} \phi$$

where  $A(u)$  is the  $n(n+3)/2$  square matrix the rows of which are  $\partial_j u$  for  $1 \leq j \leq n$  and  $\partial_j \partial_k u$  for  $1 \leq j \leq k \leq n$ ; thanks to our choice of  $u_0$ , the matrix  $A(u_0)$  is invertible on  $\Omega$ , and so is  $A(u)$  for any  $u$  close enough to  $u_0$ . Since  $A(u)^{-1}$  is an algebraic function of derivatives of  $u$  up to order 2, estimates such as (3) are again classical.

Finally, we have to prove that this operator  $\psi$  inverts  $\phi'$  (formula (2)). Applying  $A(u)$  to the function  $v$  in (16), one gets

$$\begin{aligned} \langle \partial_j u, v \rangle &= -\frac{1}{2} \phi_j = 0 & 1 \leq j \leq n \\ \langle \partial_j \partial_k u, v \rangle &= -\frac{1}{2} \phi_{jk} & 1 \leq j \leq k \leq n. \end{aligned}$$

The  $x_k$  derivative of the first equation gives  $\langle \partial_j \partial_k u, v \rangle + \langle \partial_j u, \partial_k v \rangle = 0$ , and one gets also  $\langle \partial_j \partial_k u, v \rangle + \langle \partial_k u, \partial_j v \rangle = 0$  so that the second equation and (15) give  $\phi'(u)v = \phi$  in  $\Omega$ .

Thus all the assumptions of the theorem are fulfilled, and it follows that we can get a solution if  $\phi(u_0)$  is sufficiently small in some  $H^s(\Omega)$  norm; but according to (14),  $\phi(u_0) = g^0 - g$ , and the result is that (13) can be solved for any metric  $g$  close enough to  $g^0$ , as required.

#### APPENDIX:

##### CONSTRUCTION OF THE SMOOTHING OPERATORS IN SOBOLEV SPACES

Let us recall that  $v \in H^s(\mathbf{R}^n)$  means  $v \in \mathcal{S}'(\mathbf{R}^n)$  and

$$|v|_s^2 = (2\pi)^{-n} \int (1 + |\xi|^2)^s |\hat{v}(\xi)|^2 d\xi < \infty.$$

Let  $\chi: \mathbf{R}^n \rightarrow [0, 1]$  be a  $C^\infty$  function taking the value 1 in a neighborhood of 0 and vanishing for  $|\xi| \geq \sqrt{3}$ . For  $v \in H^\infty(\mathbf{R}^n)$  and  $\theta > 1$  one sets

$$\widehat{S_\theta v}(\xi) = \chi(\xi/\theta) \hat{v}(\xi).$$

Then, if  $s \geq t$ ,

$$\begin{aligned} (1 + |\xi|^2)^s |\widehat{S_\theta v}(\xi)|^2 &\leq \theta^{2(s-t)} (1 + |\xi/\theta|^2)^{s-t} |\chi(\xi/\theta)|^2 (1 + |\xi|^2)^t |\hat{v}(\xi)|^2 \\ &\leq (2\theta)^{2(s-t)} (1 + |\xi|^2)^t |\hat{v}(\xi)|^2 \end{aligned}$$