# Application to the local isometric embedding of a Riemannian manifold

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(following Hörmander [3], Section 2).

Let $M$ be a compact $C^\infty$ manifold of dimension $n$ and $g$ a smooth Riemannian metric on $M$. In local coordinates, we are thus given a positive definite quadratic form

$$g = \sum_{j,k} g_{jk} dx_j dx_k .$$

The celebrated theorem of Nash [7], which is at the origin of the method, states that for some (large) integer $N$, there is an isometric embedding $u: M \to \mathbf{R}^N$, that is an injective map satisfying the system of equations

$$\langle \partial_j u, \partial_k u \rangle = g_{jk} \quad 1 \leqslant j, k \leqslant n \tag{13}$$

where $\partial_j$ stands for $\partial/\partial x_j$ and $\langle\ ,\ \rangle$ for the Euclidean scalar product in $\mathbf{R}^N$; thus, any compact Riemannian manifold can be thought as a submanifold of a Euclidean space.

In the proof of this Nash theorem, one first establishes that the set of metrics $g$ such that the problem can be solved is a dense convex cone in the set of all $C^\infty$ metrics on $M$, and this leads to the following reduced problem (see Hörmander [3] Section 2): show that the equation (13) can be solved for every metric in some neighborhood of a fixed metric $g^0$.

To illustrate the method described above, let us show how one can use our theorem to prove this last property locally (and this will give a local isometric embedding $u: M \to \mathbf{R}^N$).

Let $\Omega = \{x \in \mathbf{R}^n ; |x| < 1\}$ and choose, near some point $x_0 \in M$, local coordinates such that $\Omega$ describes a neighborhood of $x_0$; we take a $C_0^\infty u_0 : \mathbf{R}^n \to \mathbf{R}^{n(n+3)/2}$ equal to

$$\left((x_j)_{1 \leqslant j \leqslant n}, (x_j^2/2)_{1 \leqslant j \leqslant n}, (x_j x_k)_{1 \leqslant j < k \leqslant n}\right)$$

in a neighborhood of $\bar{\Omega}$; this $u_0$ is an isometric embedding for the corresponding metric $g^0$ in $\Omega$, namely the metric $g_{jj}^0 = 1 + |x|^2$ and $g_{jk} = x_j x_k$ if $j \neq k$. Finally, for a metric $g$ close to $g^0$, we consider the restriction $\phi(u)$ to $\Omega$ of the function

$$(\langle \partial_j u, \partial_k u \rangle - g_{jk})_{1 \leqslant j \leqslant k \leqslant n} \tag{14}$$

which is a function in $H^\infty(\Omega)$ valued in $\mathbf{R}^{n(n+1)/2}$ for any $u \in H^\infty(\mathbf{R}^n)$ valued in $\mathbf{R}^{n(n+3)/2}$. Classically, estimates such as (1) hold for $s > (n+2)/2$.

The derivative of $\phi$ with respect to $u$ is defined by

$$\phi'(u)v = (\langle \partial_j u, \partial_k v \rangle + \langle \partial_k u, \partial_j v \rangle)_{1 \leqslant j \leqslant k \leqslant n} . \tag{15}$$

If $\phi \in H^{\infty}(\Omega)$ is valued in $\mathbf{R}^{n(n+1)/2}$, let us consider it as a function valued in $\mathbf{R}^{n(n+3)/2}$ by adding $n$ components $\varphi_j = 0$ for $1 \leqslant j \leqslant n$, and define $\psi(u)\varphi$ as a continuous extension to $\mathbf{R}^n$ of the function

$$v = -\frac{1}{2} A(u)^{-1}\varphi \tag{16}$$

where $A(u)$ is the $n(n+3)/2$ square matrix the rows of which are $\partial_j u$ for $1 \leqslant j \leqslant n$ and $\partial_j \partial_k u$ for $1 \leqslant j \leqslant k \leqslant n$; thanks to our choice of $u_0$, the matrix $A(u_0)$ is invertible on $\Omega$, and so is $A(u)$ for any $u$ close enough to $u_0$. Since $A(u)^{-1}$ is an algebraic function of derivatives of $u$ up to order 2, estimates such as (3) are again classical.

Finally, we have to prove that this operator $\psi$ inverts $\phi'$ (formula (2)). Applying $A(u)$ to the function $v$ in (16), one gets

$$\langle \partial_j u, v \rangle = -\frac{1}{2}\varphi_j = 0 \qquad 1 \leqslant j \leqslant n$$

$$\langle \partial_j \partial_k u, v \rangle = -\frac{1}{2}\varphi_{jk} \qquad 1 \leqslant j \leqslant k \leqslant n\,.$$

The $x_k$ derivative of the first equation gives $\langle \partial_j \partial_k u, v \rangle + \langle \partial_j u, \partial_k v \rangle = 0$, and one gets also $\langle \partial_j \partial_k u, v \rangle + \langle \partial_k u, \partial_j v \rangle = 0$ so that the second equation and (15) give $\phi'(u)v = \varphi$ in $\Omega$.

Thus all the assumptions of the theorem are fulfilled, and it follows that we can get a solution if $\phi(u_0)$ is sufficiently small in some $H^s(\Omega)$ norm; but according to (14), $\phi(u_0) = g^0 - g$, and the result is that (13) can be solved for any metric $g$ close enough to $g^0$, as required.

## APPENDIX:

## CONSTRUCTION OF THE SMOOTHING OPERATORS IN SOBOLEV SPACES

Let us recall that $v \in H^s(\mathbf{R}^n)$ means $v \in \mathcal{S}'(\mathbf{R}^n)$ and

$$|v|_s^2 = (2\pi)^{-n} \int (1+|\xi|^2)^s\, |\hat{u}(\xi)|^2 d\xi < \infty\,.$$

Let $\chi : \mathbf{R}^n \to [0, 1]$ be a $C^\infty$ function taking the value 1 in a neighborhood of 0 and vanishing for $|\xi| \geqslant \sqrt{3}$. For $v \in H^\infty(\mathbf{R}^n)$ and $\theta > 1$ one sets

$$\widehat{S_\theta v}(\xi) = \chi(\xi/\theta)\hat{v}(\xi)\,.$$

Then, if $s \geqslant t$,

$$(1+|\xi|^2)^s\, |\widehat{S_\theta v}(\xi)|^2 \leqslant \theta^{2(s-t)}(1+|\xi/\theta|^2)^{s-t}\,|\chi(\xi/\theta)|^2(1+|\xi|^2)^t\,|\hat{v}(\xi)|^2$$
$$\leqslant (2\theta)^{2(s-t)}(1+|\xi|^2)^t\,|\hat{v}(\xi)|^2$$