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**Artikel:** SIMPLE NASH-MOSER IMPLICIT FUNCTION THEOREM  
**Anhang:** Appendix : Construction of the Smoothing Operators in Sobolev Spaces  
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If  $\phi \in H^\infty(\Omega)$  is valued in  $\mathbf{R}^{n(n+1)/2}$ , let us consider it as a function valued in  $\mathbf{R}^{n(n+3)/2}$  by adding  $n$  components  $\phi_j = 0$  for  $1 \leq j \leq n$ , and define  $\psi(u)\phi$  as a continuous extension to  $\mathbf{R}^n$  of the function

$$(16) \quad v = -\frac{1}{2} A(u)^{-1} \phi$$

where  $A(u)$  is the  $n(n+3)/2$  square matrix the rows of which are  $\partial_j u$  for  $1 \leq j \leq n$  and  $\partial_j \partial_k u$  for  $1 \leq j \leq k \leq n$ ; thanks to our choice of  $u_0$ , the matrix  $A(u_0)$  is invertible on  $\Omega$ , and so is  $A(u)$  for any  $u$  close enough to  $u_0$ . Since  $A(u)^{-1}$  is an algebraic function of derivatives of  $u$  up to order 2, estimates such as (3) are again classical.

Finally, we have to prove that this operator  $\psi$  inverts  $\phi'$  (formula (2)). Applying  $A(u)$  to the function  $v$  in (16), one gets

$$\begin{aligned} \langle \partial_j u, v \rangle &= -\frac{1}{2} \phi_j = 0 & 1 \leq j \leq n \\ \langle \partial_j \partial_k u, v \rangle &= -\frac{1}{2} \phi_{jk} & 1 \leq j \leq k \leq n. \end{aligned}$$

The  $x_k$  derivative of the first equation gives  $\langle \partial_j \partial_k u, v \rangle + \langle \partial_j u, \partial_k v \rangle = 0$ , and one gets also  $\langle \partial_j \partial_k u, v \rangle + \langle \partial_k u, \partial_j v \rangle = 0$  so that the second equation and (15) give  $\phi'(u)v = \phi$  in  $\Omega$ .

Thus all the assumptions of the theorem are fulfilled, and it follows that we can get a solution if  $\phi(u_0)$  is sufficiently small in some  $H^s(\Omega)$  norm; but according to (14),  $\phi(u_0) = g^0 - g$ , and the result is that (13) can be solved for any metric  $g$  close enough to  $g^0$ , as required.

#### APPENDIX:

##### CONSTRUCTION OF THE SMOOTHING OPERATORS IN SOBOLEV SPACES

Let us recall that  $v \in H^s(\mathbf{R}^n)$  means  $v \in \mathcal{S}'(\mathbf{R}^n)$  and

$$|v|_s^2 = (2\pi)^{-n} \int (1 + |\xi|^2)^s |\hat{v}(\xi)|^2 d\xi < \infty.$$

Let  $\chi: \mathbf{R}^n \rightarrow [0, 1]$  be a  $C^\infty$  function taking the value 1 in a neighborhood of 0 and vanishing for  $|\xi| \geq \sqrt{3}$ . For  $v \in H^\infty(\mathbf{R}^n)$  and  $\theta > 1$  one sets

$$\widehat{S_\theta v}(\xi) = \chi(\xi/\theta) \hat{v}(\xi).$$

Then, if  $s \geq t$ ,

$$\begin{aligned} (1 + |\xi|^2)^s |\widehat{S_\theta v}(\xi)|^2 &\leq \theta^{2(s-t)} (1 + |\xi/\theta|^2)^{s-t} |\chi(\xi/\theta)|^2 (1 + |\xi|^2)^t |\hat{v}(\xi)|^2 \\ &\leq (2\theta)^{2(s-t)} (1 + |\xi|^2)^t |\hat{v}(\xi)|^2 \end{aligned}$$

since  $|\chi| \leq 1$  and  $|\xi/\theta| \leq \sqrt{3}$  for  $(\xi/\theta) \in \text{supp } \chi$ ; this gives the first estimate (4) with  $C_{s,t} = 2^{s-t}$ .

Similarly, for  $s \leq t$ ,

$$(1 + |\xi|^2)^s |\hat{v}(\xi) - \widehat{S_\theta v}(\xi)|^2 = |1 - \chi(\xi/\theta)|^2 (1 + |\xi|^2)^s |\hat{v}(\xi)|^2;$$

a Taylor formula gives  $|1 - \chi(\xi/\theta)| \leq C_k |\xi/\theta|^k$  with  $C_k = \sup |\chi^{(k)}|/k!$  for any  $k \in \mathbb{N}$  since  $\chi(0) = 1$  and  $\chi^{(j)}(0) = 0$  for  $j > 0$ , so that for  $t = s + k$

$$\begin{aligned} (1 + |\xi|^2)^s |\hat{v}(\xi) - \widehat{S_\theta v}(\xi)|^2 &\leq C_{t-s}^2 |\xi/\theta|^{2(t-s)} (1 + |\xi|^2)^s |\hat{v}(\xi)|^2 \\ &\leq C_{t-s}^2 \theta^{2(s-t)} (1 + |\xi|^2)^t |\hat{v}(\xi)|^2 \end{aligned}$$

whence the second estimate (4) with  $C_{s,t} = C_{t-s} = \sup |\chi^{(t-s)}|/(t-s)!$

#### REFERENCES

- [1] HAMILTON, R. The inverse function theorem of Nash-Moser. *Bulletin of the A.M.S.* 7 (1982), 65-222.
- [2] HÖRMANDER, L. The boundary problems of physical geodesy. *Arch. Rat. Mech. Anal.* 62 (1976), 1-52.
- [3] ——— *Implicit function theorems*. Lectures at Stanford University, Summer Quarter 1977.
- [4] ——— On the Nash-Moser implicit function theorem. *Annales Acad. Sci. Fenniae, Series A.I. Math.* 10 (1985), 255-259.
- [5] MOSER, J. A new technique for the construction of solutions of nonlinear differential equations. *Proc. Nat. Acad. Sci.* 47 (1961), 1824-1831.
- [6] ——— A rapidly convergent iteration method and nonlinear partial differential equations I and II. *Ann. Scuola Norm. Sup. di Pisa* 20 (1966), 265-315 and 499-533.
- [7] NASH, J. The imbedding problem for Riemannian manifolds. *Ann. of Math.* 63 (1956), 20-63.
- [8] SCHWARTZ, J. T. *Nonlinear functional analysis, Chap. II.A*. Gordon & Breach, New York 1969.
- [9] SERGERAERT, F. Une généralisation du théorème des fonctions implicites de Nash. *C. R. Acad. Sci. Paris, 270A* (1970), 861-863.
- [10] ——— Un théorème des fonctions implicites sur certains espaces de Fréchet et quelques applications. *Ann. Sci. Ec. Norm. Sup. Paris 4<sup>e</sup> série*, 5 (1972), 599-660.
- [11] ZEHNDER, E. Generalized implicit function theorems with applications to some small divisor problems I and II. *Comm. in Pure and Appl. Math.* 28 (1975), 91-140; 29 (1976), 49-111.

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