COMPOSITION PRODUCTS AND MODELS FOR THE HOMFLY POLYNOMIAL

Autor(en): **Jaeger, François**

Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **35 (1989)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-57379

L'Enseignement Mathématique, t. 35 (1989), p. 323-361

COMPOSITION PRODUCTS AND MODELS FOR THE HOMFLY POLYNOMIAL

by François JAEGER

ABSTRACT: We define a composition product for homfly polynomials of oriented links and we show how this operation can be used to construct in a natural way a sequence of state models due to Jones. We also present a refinement of this result in the case of closed braids. This leads us first to a new state model for the Alexander-Conway polynomial which can be interpreted as an ice-type model. Then we express the homfly polynomial of a braid diagram in terms of the Alexander-Conway polynomials of its subdiagrams. As a consequence, we obtain simple direct proofs of inequalities due to Morton, Franks and Williams. Finally we give a state model for the homfly polynomial of a closed braid.

RÉSUMÉ: Nous définissons un produit de composition pour les polynômes homfly des entrelacs orientés et nous montrons comment on peut utiliser cette opération pour construire de façon naturelle une suite de « modèles d'états » due à Jones. Nous présentons également un raffinement de ce résultat dans le cas des tresses fermées. Ceci nous conduit d'abord à un nouveau modèle d'états pour le polynôme d'Alexander-Conway qui peut s'interpréter comme un modèle « de type glace ». Puis nous exprimons le polynôme homfly d'un diagramme de tresse en termes des polynômes d'Alexander-Conway de ses sous-diagrammes. Comme conséquence, nous obtenons des preuves simples et directes d'inégalités dues à Morton, Franks et Williams. Enfin nous donnons un modèle d'états pour le polynôme homfly d'une tresse fermée.

1. INTRODUCTION

Since its discovery [1], the Alexander polynomial has played an important role in the development of knot theory. Its topological and algebraic aspects (relations with the fundamental group and the infinite cyclic cover of the complement) have been extensively studied. Its combinatorial aspects, already present in the original paper by Alexander, have recently received new attention. The starting point was Conway's work [3] which showed how a suitable normalization of the Alexander polynomial of an oriented link (which we shall call the Alexander-Conway polynomial) can be computed recursively on an arbitrary regular projection — or "diagram" — of the link by using a linear equation satisfied by the values of the polynomial on three links which are "skein related" (that is, they are represented by diagrams D^+, D^-, D° which differ only inside a small disk where they behave as depicted on Figure 3). In fact the Alexander-Conway polynomial can be described in purely combinatorial terms, as shown by Kauffman in [16]: to each diagram is associated in a simple way a polynomial in one variable which is shown to satisfy Conway's "skein equation" and to be invariant under Reidemeister moves. The polynomial is defined as a summation over a set of possible "states" of the diagram and can be viewed as the partition function of a certain model (we shall call it a "state model") in the sense of Statistical Mechanics [2].

More recently Jones, in relation with his work on Von Neumann algebras, discovered another one-variable polynomial link invariant [11, 12] which like the Alexander-Conway polynomial satisfies a skein equation. Both invariants were soon generalized by different authors [5, 26] into a two-variable polynomial which became known as the Jones-Conway or "homfly" (from the initials of the authors of [5]) polynomial. The homfly polynomial (as we have arbitrarily chosen to call it) can be defined combinatorially on diagrams, in relation with the Conway-type algorithm which allows its computation. It can also be defined via representations of Artin's braid groups in Hecke algebras, using theorems of Alexander (which asserts that every oriented link can be represented as a closed braid) and Markov (which characterizes the isotopy of closed braids). In both cases the proof of the existence of the homfly polynomial is quite sophisticated.

Kauffman [17] obtained an elegant and simple state model for the Jones polynomial which has lead to the solution of old conjectures on alternating links [17, 24, 27]. Jones ([14], see also [20] and [28]) also obtained state models for an infinite sequence of one-variable specializations of the homfly polynomial. For a rather special kind of diagram the homfly polynomial is equivalent to the Tutte polynomial of an associated plane graph [10], which has simple and well-known state models [2, 9, 29, 30]. However no state model is known for the full homfly polynomial of an arbitrary diagram. Moreover no natural topological 3-dimensional interpretation of the homfly polynomial has been found, apart from the case of the Alexander-Conway polynomial. The purpose of this paper is to present some progress towards the solution of these problems.

In Section 2 we introduce a composition product for homfly polynomials. This product allows the combinatorial definition of the homfly polynomial of a diagram for a given pair of values of the variables in terms of the homfly polynomials of its subdiagrams for other related pairs of values of the variables (Proposition 1). We show in Proposition 2 how the sequence of state models due to Jones can be derived simply from the product operation, starting from an elementary special case of the homfly polynomial. Then, motivated by some difficulties in the application of the concept of composition product to the Alexander-Conway polynomial, in Section 3 we restrict our attention to closed braids and we introduce a specified composition product for this class of diagrams (Proposition 3). This leads us first to another version of the Jones sequence of state models (Proposition 4). Then we obtain a state model for the Alexander-Conway polynomial (Proposition 6) which can be interpreted as an ice-type model (Proposition 7). As another consequence we give an expansion of the homfly polynomial of a braid diagram in terms of the Alexander-Conway polynomials of its subdiagrams (Proposition 9). This yields simple direct proofs of some inequalities due to Morton [22] and independently Franks and Williams [4] which have been helpful in the study of the braid index. Finally we combine the previous results to obtain a state model for the homfly polynomial of a closed braid (Proposition 12). We present some perspectives for further research in Section 4.

2. The composition product of homfly polynomials

2.1. Definitions

By *diagram* we mean a regular plane projection of a tame oriented link in 3-space. We shall consider diagrams as 4-regular directed plane graphs.

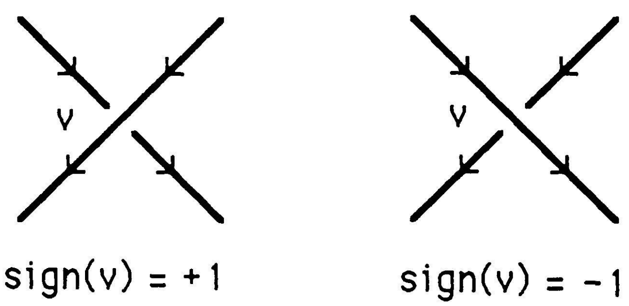

Figure 1

In particular a simple Jordan curve (corresponding to the trivial knot) is a graph with no vertices and one edge which we call the *free loop*. The vertices of a diagram will be signed according to the convention described in Figure 1.

The *writhe* of the diagram D, denoted by $w(D)$, is the sum of the signs of the vertices of D. We define the *rotation number* of D, denoted by $r(D)$, as the sum of the signs of the Seifert circles of D, where the sign of such a circle is 1 if it is oriented counterclockwise and -1 otherwise (this combinatorial form of the Whitney degree appears in [16] p. 95-100, where it is called *curliness*).

Two diagrams will be said to be *isotopic* if they represent the same oriented link up to ambient isotopy. We shall need the following economical form of Reidemeister's Theorem given in [28]: two diagrams are isotopic if and only if one can be obtained from the other by a finite sequence of moves of types A_i, A'_i, B_i, B'_i $(i=1, 2, 3, 4)$ and C, C' described in Figure 2.

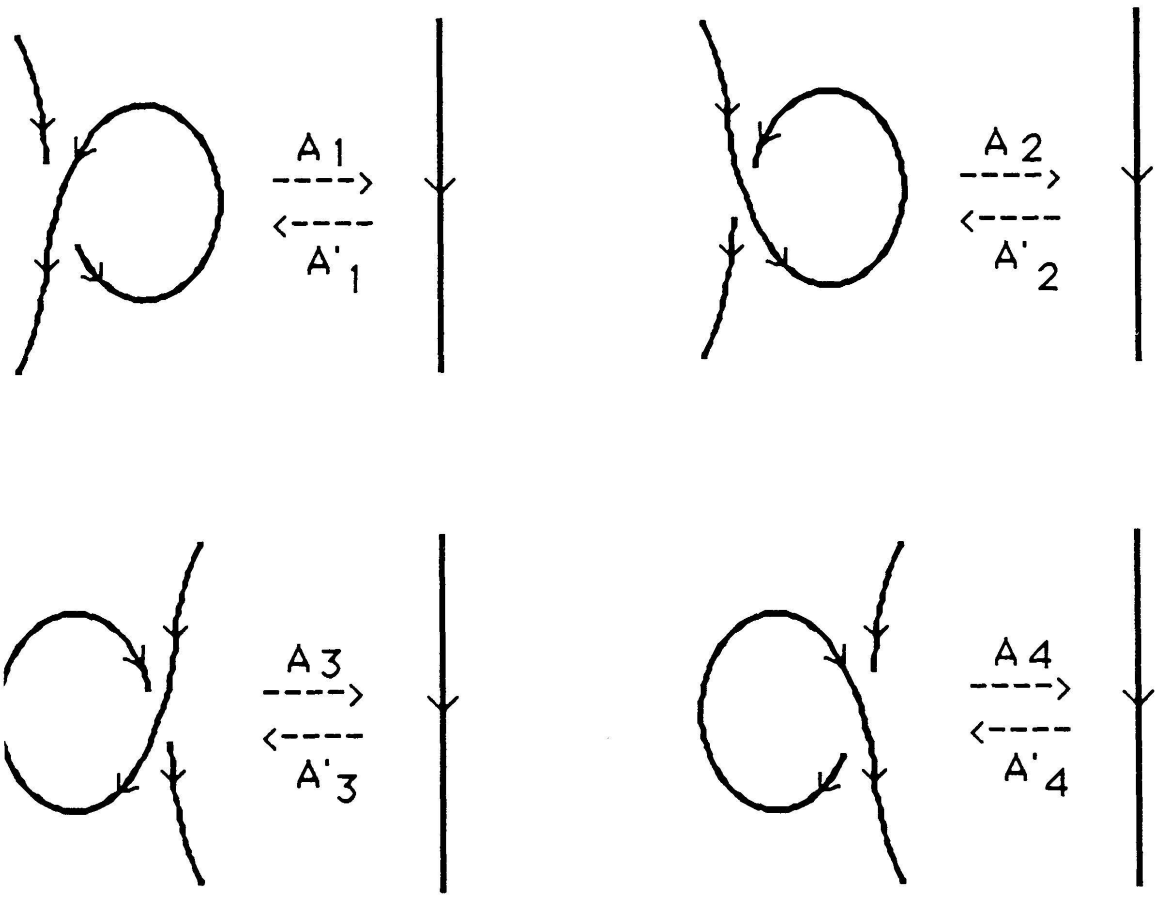

FIGURE 2A

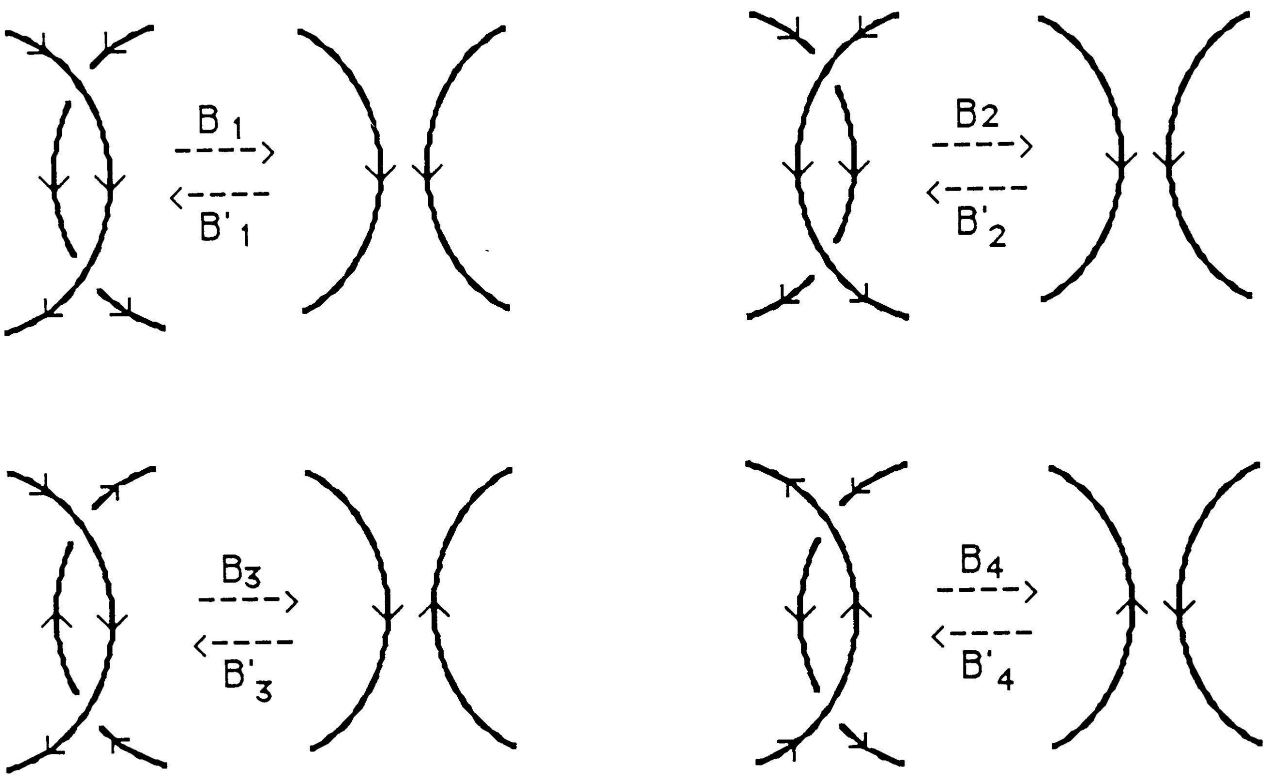

FIGURE 2B

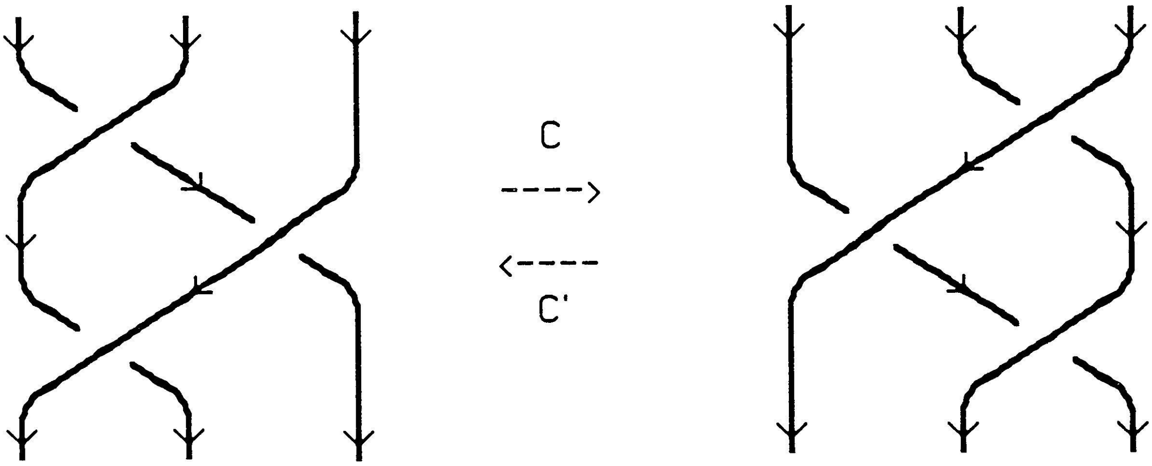

FIGURE 2C

Two diagrams will be said to be *regularly isotopic* if one can be obtained from the other by a finite sequence of moves of types $B_i, B'_i\,(i=1, 2, 3, 4)$ and C, C' (this concept is due to Kauffman [19]). The writhe and the rotation number are invariants of regular isotopy: if D and D' are regularly isotopic diagrams, then $w(D) = w(D')$ (this is immediate) and $r(D) = r(D')$ (see [16] p. 95-100).

If D^+, D^- and D° are diagrams which are identical outside a small disk and behave as depicted in Figure 3 inside that disk, we shall say that (D^+, D^-, D°) form a *Conway triple*.

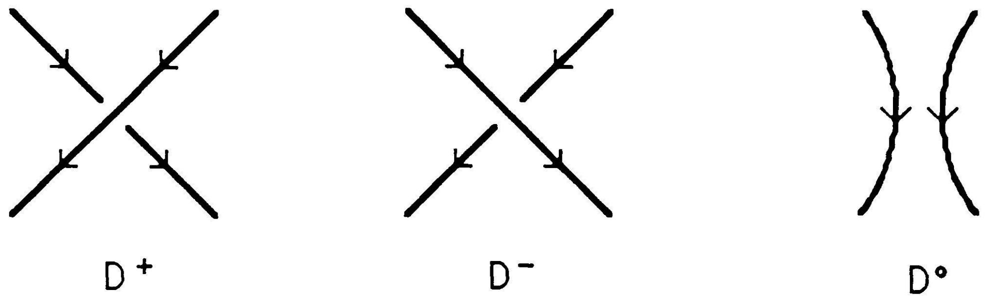

FIGURE 3

We shall be concerned here with the following result:

THEOREM. *One can associate (in an unique way) to every diagram* D *a Laurent polynomial with integer coefficients in two variables* z, a *which we denote by* $H(D, z, a)$ *in such a way that the following properties hold:*

(i) *If* D *and* D' *are regularly isotopic,* $H(D, z, a) = H(D', z, a)$.

(ii) *If* D' *is obtained from* D *by a move of type* A'_1 *or* A'_3 *(respectively:* A'_2 *or* A'_4*) then* $H(D', z, a) = aH(D, z, a)$ *(respectively:* $H(D', z, a) = a^{-1}H(D, z, a)$*).*

(iii) *If* (D^+, D^-, D°) *form a Conway triple then:*

$$H(D^+, z, a) - H(D^-, z, a) = zH(D^\circ, z, a)\,.$$

(iv) *If* D *is the free loop,* $H(D, z, a) = 1$.

Then, as observed in [19], if we set $P(D, z, a) = a^{-w(D)}H(D, z, a)$ this defines the following version of the homfly polynomial [5, 6, 8, 21, 25, 26]: P is an isotopy invariant which takes the value 1 on the free loop and satisfies, for every Conway triple (D^+, D^-, D°): $aP(D^+, z, a) - a^{-1}P(D^-, z, a) = zP(D^\circ, z, a)$. In particular, $P(D, z, 1) = H(D, z, 1)$ is the Alexander-Conway polynomial of D [1, 3, 15, 16] and we denote it by $A(D, z)$.

We shall need the following easy consequence of the above Theorem:

(v) If the diagram D' is obtained from the diagram D by the addition of a single free loop, then $H(D', z, a) = (a - a^{-1})z^{-1}H(D, z, a)$.

For the sake of simplicity we define: $H'(D, z, a) = (a - a^{-1})z^{-1}H(D, z, a)$. Thus H' can replace H in properties (i), (ii), (iii) of the Theorem, and satisfies

(iv′) If D is the free loop, $H'(D, z, a) = (a - a^{-1})z^{-1}$.

In the sequel we shall have to consider the *empty diagram* ω, which has no vertices and no edges. It will be convenient to set $H'(\omega, z, a) = 1$, so that property (v) is valid with H' instead of H even when D is the empty diagram. This convention together with property (v) can replace property (iv') in the definition of H'.

2.2. Labellings and the composition product

We define a *labelling* of a diagram D as a mapping f from the edge-set of D to the set of positive integers which satisfies the following

conservation law: for every positive integer i, at every vertex v of D, the number of edges labelled i (that is, edges in $f^{-1}(i)$) incident towards v equals the number of such edges incident from v (a loop at v contributing 1 to both numbers).

Then if we first erase all edges not labelled i and the isolated vertices thus created, "smoothing out" all vertices of degree 2 (see Figure 4) and retaining the signs (or equivalently the crossing structure) at every vertex of degree 4, we obtain a (possibly empty) diagram which we denote by $D_{f,i}$ and call a *subdiagram* of D. We may associate to every edge e of D with $f(e) = i$ a unique simple (possibly closed) directed path in D containing e which is converted by the above process into an edge of $D_{f,i}$. This edge of $D_{f,i}$ will be denoted by $P_f(e)$. Thus we have defined a mapping P_f from the edge-set of D to the union of the edge-sets of all $D_{f,i}$. We call this mapping P_f the *projection associated to* f.

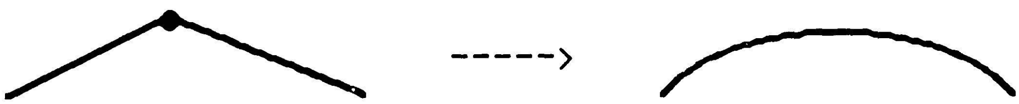

Figure 4

For any labelling f of the diagram D, we may write $r(D) = \sum_i r(D_{f,i})$ (with the obvious convention that an empty diagram contributes zero to this sum). This additivity property of the rotation number is immediate from the definition of this number as a Whitney degree (see [16] p. 95-100) and we shall use it implicitly in the sequel.

We define the *interaction* $<v \mid D \mid f>$ of the vertex v in the diagram D with the labelling f as follows. If the edges incident to v are assigned only one label, or two distinct labels i and j in such a way that $D_{f,i}$ and $D_{f,j}$ cross at v, then $<v \mid D \mid f> = 1$. Otherwise $<v \mid D \mid f>$ is defined on Figure 5. If W is some set of vertices of D we write $<W \mid D \mid f> = \prod_{v \in W} <v \mid D \mid f>$. We shall take $<W \mid D \mid f>$ as equal to 1 if W is empty. We write more briefly $<D \mid f>$ for $<V \mid D \mid f>$ if D has vertex-set V.

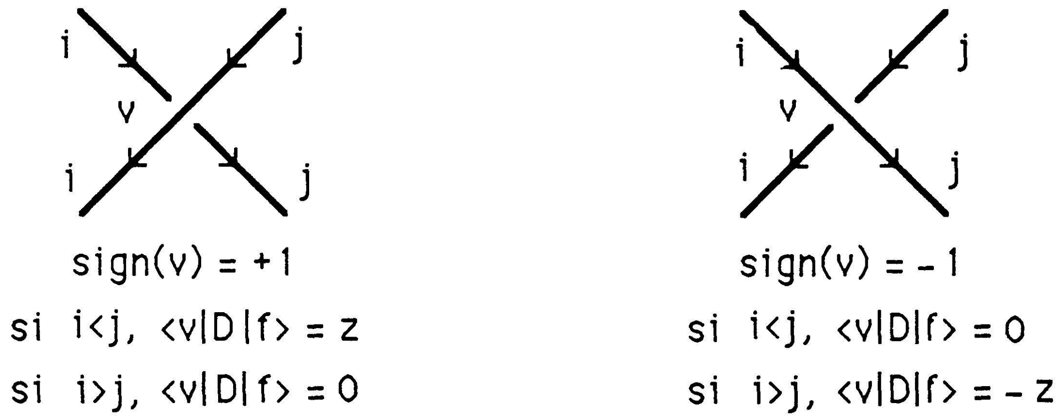

FIGURE 5

We denote by $L(D, k)$ the set of labellings of D which take their values in $\{1, ..., k\}$.

PROPOSITION 1. *For any diagram* D,

$$\sum_{f \in L(D,2)} < D \mid f > a_2^{-r(D_{f,1})} a_1^{r(D_{f,2})} H'(D_{f,1}, z, a_1) H'(D_{f,2}, z, a_2) = H'(D, z, a_1 a_2).$$

Proof. Let us write $H''(D, z, a_1, a_2) = \sum_{f \in L(D,2)} C(D, f)$ with

$$C(D, f) = < D \mid f > a_2^{-r(D_{f,1})} a_1^{r(D_{f,2})} H'(D_{f,1}, z, a_1) H'(D_{f,2}, z, a_2).$$

We shall show that the expression $H''(D, z, a_1, a_2)$ satisfies properties (i), (ii), (iii), (v) of Section 2.1 with $a = a_1 a_2$. We now introduce the general method which will be used in the different cases.

Consider a pair of diagrams (D, D') which are identical outside some disk and take specified forms inside this disk. This is the case when D and D' are related by a Reidemeister move (Figure 2 specifies the forms inside the disk). Similarly if D' is obtained from D by the addition of a free loop with empty interior, we may consider a disk which is empty in D and contains this free loop in D', so that D and D' are identical outside that disk. We shall call such a disk a *separator* and D and D' will be said to be *compatible* with respect to this separator. Similarly a disk involved in the definition of a Conway triple will also be called a separator with respect to which the elements of the triple are compatible.

Consider now a diagram D with a given separator S. Vertices of D situated in the interior (respectively: exterior) of S will be called *inner* (respectively: *outer*), and we assume that there are no other vertices. An edge of D which meets the exterior of S will also be called *outer*. If we shrink S into a new vertex s we obtain a plane graph which we call the *outerdiagram* associated to (D, S). An edge of the outerdiagram which is incident to s will be called a *boundary edge*. Such an edge e can be identified with a portion of some outer edge of D which crosses the boundary of S and we shall denote this unique outer edge by $o(e)$. Similarly an edge e of the outerdiagram which is not a boundary edge can be identified with a unique outer edge of D also denoted by $o(e)$.

We call *outer labelling* of D (with respect to S) a mapping from the set of edges of the associated outerdiagram to the set of positive integers which satisfies the conservation law at every outer vertex. Then clearly the conservation law also holds at the special vertex s. We denote the set of outer labellings of D with values in $\{1, 2\}$ by $L^\circ(D, 2)$. For f in $L^\circ(D, 2)$ and g in $L(D, 2)$ we write $f \subseteq g$ to indicate that f can be obtained from g by "labelled shrinking", in other words that for every edge e of the outerdiagram, $f(e) = g(o(e))$.

Now we may write: $H''(D, z, a_1, a_2) = \sum_{f \in L^\circ(D, 2)} C(D, f)$, with

$$C(D, f) = \sum_{g \in L(D, 2),\, f \subseteq g} C(D, g)\,.$$

The properties (i), (ii), (iii), (v) to be proved take the following form:

$$\sum_i x_i H''(D_i, z, a_1, a_2) = 0$$

for some family of diagrams (a pair or a triple) (D_i) compatible with respect to a separator S. Thus all diagrams of this family have the same set of outer labellings with values in $\{1, 2\}$.

We shall show that for every such outer labelling f: $\sum_i x_i C(D_i, f) = 0$.

For this purpose we introduce a *reference* consisting of a diagram R compatible with the D_i (with respect to S) together with a labelling h in $L(R, 2)$ with $f \subseteq h$. Then evaluating $C = \left(\sum_i x_i C(D_i, f)\right)/C(R, h)$ instead of $\sum_i x_i C(D_i, f)$ will yield substantial simplification.

To be more precise, recall that for every diagram D in the family $((D_i), R)$ and every $g \in L(D, 2)$ with $f \subseteq g$:

$$C(D, g) = \langle D \mid g \rangle a_2^{-r(D_{g,1})} a_1^{r(D_{g,2})} H'(D_{g,1}, z, a_1) H'(D_{g,2}, z, a_2)$$

Then, denoting by $V^\circ(D)$ the set of outer vertices and by $V^i(D)$ the set of inner vertices of D, we have $\langle D \mid g \rangle = \langle V^\circ(D) \mid D \mid g \rangle \ \langle V^i(D) \mid D \mid g \rangle$. Clearly $\langle V^\circ(D) \mid D \mid g \rangle$ does not depend on the choice of D in $((D_i), R)$ and g in $L(D, 2)$ with $f \subseteq g$, and we may denote it by $\langle f \rangle$.

If $\langle f \rangle = 0$ clearly $\sum_i x_i C(D_i, f) = 0$ and we are done. In the sequel we consider only outer labellings f such that $\langle f \rangle \neq 0$. Then in evaluating C

we shall divide all contributions by $<f>$, which amounts to the replacement of the interaction $<D \mid g>$ by the *inner interaction* $<V^i(D) \mid D \mid g>$. We shall always choose R and h in such a way that $<V^i(R) \mid R \mid h> \neq 0$.

Then for every diagram D in the family (D_i) and every $g \in L(D, 2)$ with $f \subseteq g$ we shall write $C'(D, g) = C(D, g)/C(R, h)$ as an ordered product $T_1 . T_2 . T_3 . T_4 . T_5$ with

$$T_1 = <V^i(D) \mid D \mid g>/<V^i(R) \mid R \mid h> ; \; T_2 = a_2^{-r(D_{g,1})+r(R_{h,1})};$$
$$T_3 = a_1^{r(D_{g,2})-r(R_{h,2})}; \; T_4 = H'(D_{g,1}, z, a_1)/H'(R_{h,1}, z, a_1);$$
$$T_5 = H'(D_{g,2}, z, a_2)/H'(R_{h,2}, z, a_2) .$$

If $T_1 = 0$ the other terms will not be evaluated. We shall denote by $C'(D, f)$ the sum

$$\sum_{g \in L(D,2),\, f \subseteq g} C'(D, g) = C(D, f)/C(R, h) .$$

Proof of property (v). Let the diagram D' be obtained from the diagram D by the addition of a single free loop O. In order to use the geometric concept of separator as defined above, we assume that the interior of O is empty (otherwise the proof would be essentially the same. Let $r(O) = \varepsilon \in \{+1, -1\}$. The outerdiagram is D (together with a new isolated vertex s) and we consider the given outer labelling f as a labelling of D. We take as a reference $R = D$ and $h = f$.

We must show that $C'(D', f) = (a_1a_2-(a_1a_2)^{-1})z^{-1}$.

This is done on Figure 6 which displays the contributions

$$C'(D', g) \, (g \in L(D', 2),\, f \subseteq g) ,$$

written as ordered products $T_1 . T_2 . T_3 . T_4 . T_5$ as specified above and evaluated using property (v) for H'

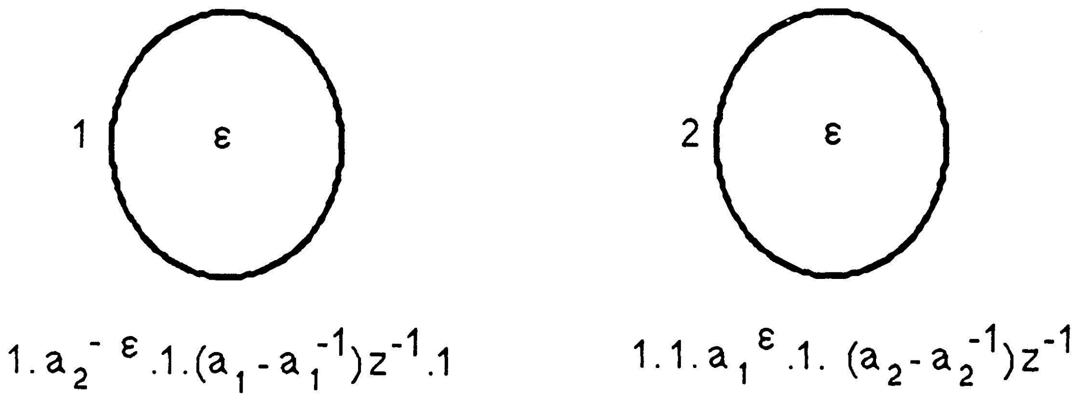

FIGURE 6

Note that the above proof also works when D is empty, so that property (iv′) indeed holds for H'' with $a = a_1a_2$.

Proof of property (iii). Let (D^+, D^-, D°) be a Conway triple and f be a given outer labelling.

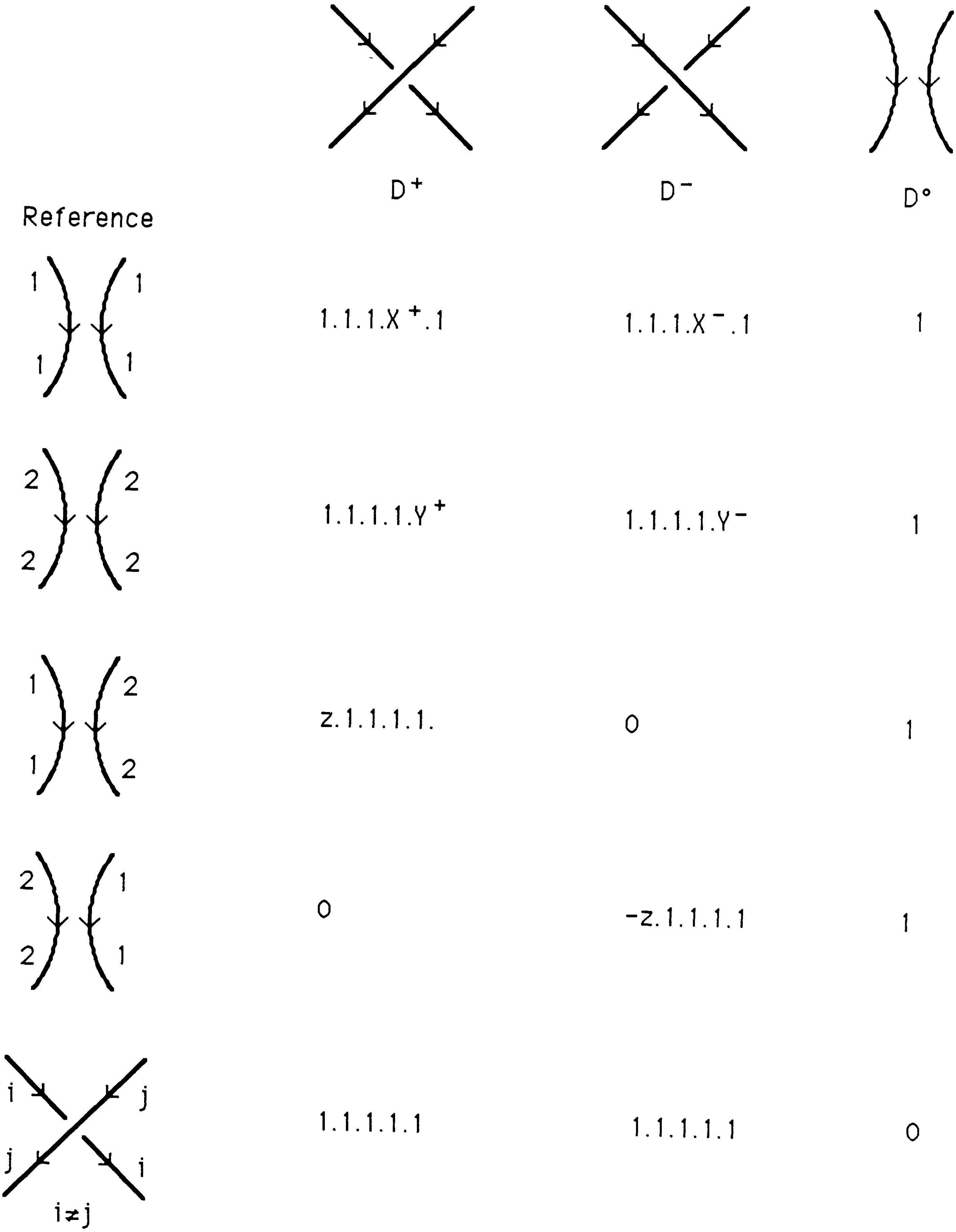

FIGURE 7

We must show that $C'(D^+, f) - C'(D^-, f) = zC'(D^\circ, f)$. The proof is given on Figure 7 which lists the various contributions. There are six cases to consider according to the labels of the boundary edges. These labels are shown on the picture representing the part of the reference diagram situated inside the separator. The labellings of all diagrams are determined uniquely by the outer labellings and are not described. In the first case, the reference is D° and X^+ denotes $H'(D^+_{g,1}, z, a_1)/H'(D^\circ_{h,1}, z, a_1)$ for the unique element g of $L(D^+, 2)$ such that $f \subseteq g$. X^- is defined similarly and the equality $X^+ - X^- = z$ follows from property (iii) for H'. The second case is settled in exactly the same way. The third and the fourth case are immediate. For the remaining two cases we note that $C'(D^\circ, f) = 0$ because there is no element g of $L(D^\circ, 2)$ such that $f \subseteq g$.

Proof of property (ii). We first observe that it is enough to consider moves of type A_1 and A_3. The move of type A_2 can be reduced to the move of type A_1, as proved diagrammatically on Figure 8, using properties (iii), (v) which have already been established. Here as usual we depict only the portions of diagrams where modifications occur, and each diagram D stands for $H''(D, z, a_1, a_2)$ (we write $a = a_1 a_2$). The proof of the reduction of the move of type A_4 to the move of type A_3 will be obtained by reversing all arrows on Figure 8.

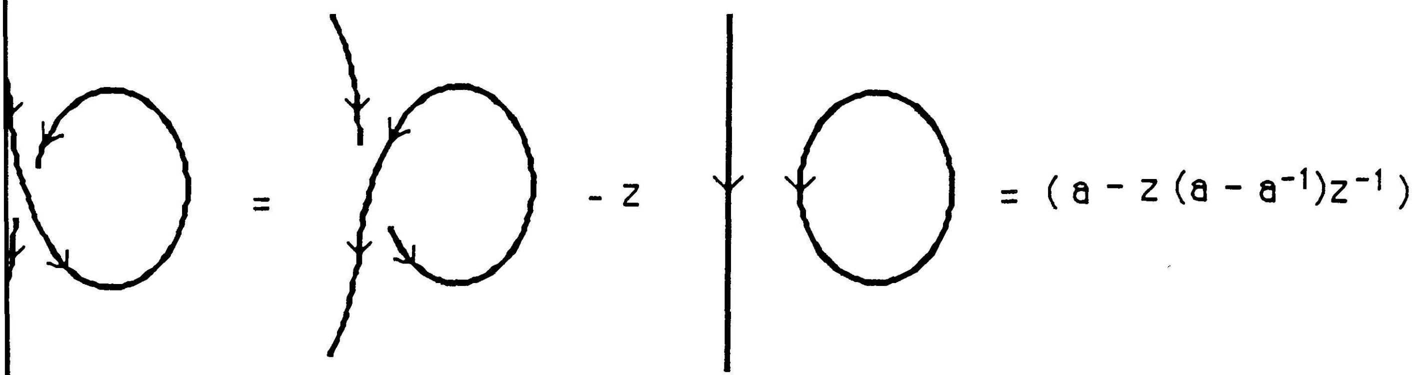

FIGURE 8

Now let D' be obtained from D by a move of type A'_1 or A'_3 (see Figure 2A), and consider an outer labelling f. We take as a reference $R = D$ and the unique element h of $L(D, 2)$ such that $f \subseteq h$. We must show that $C'(D', f) = a_1 a_2$.

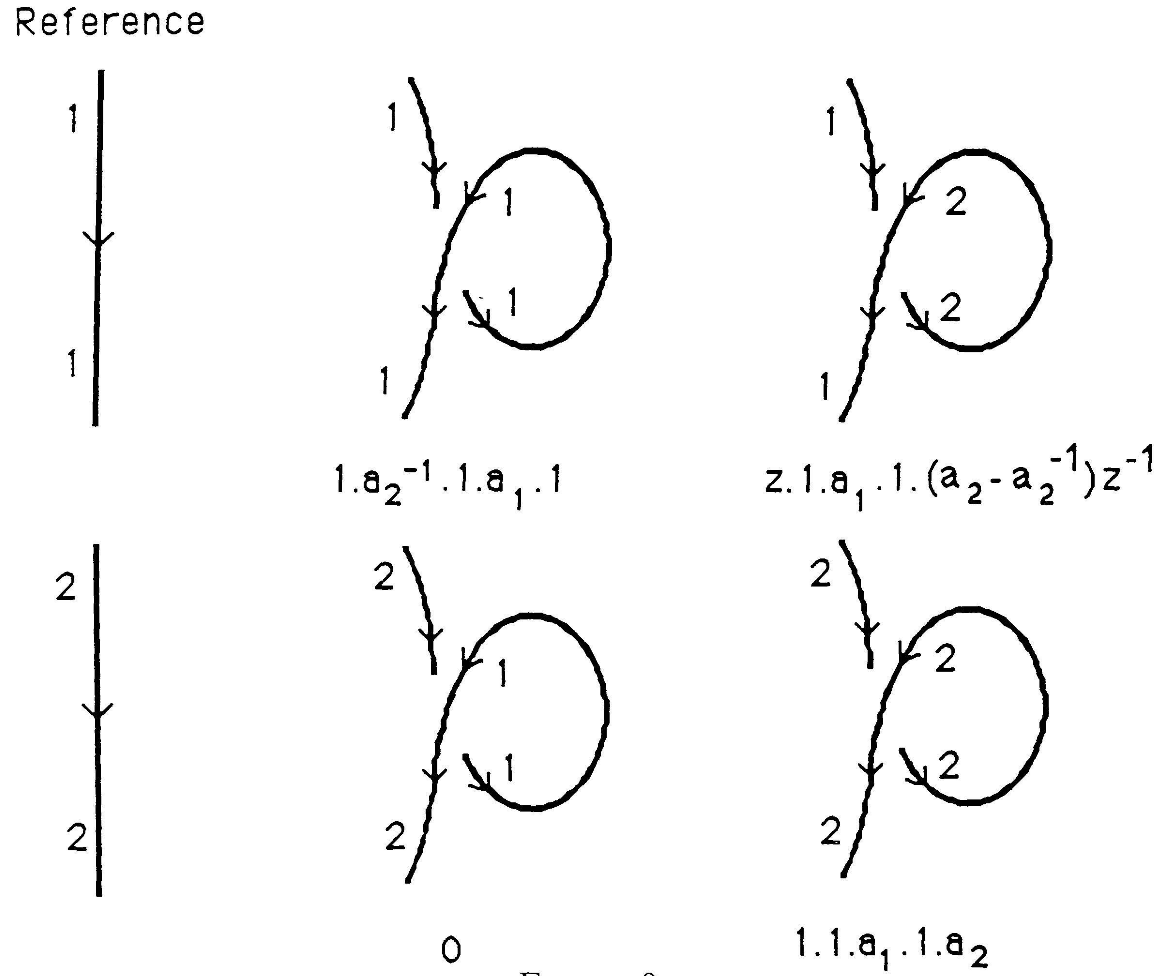

FIGURE 9

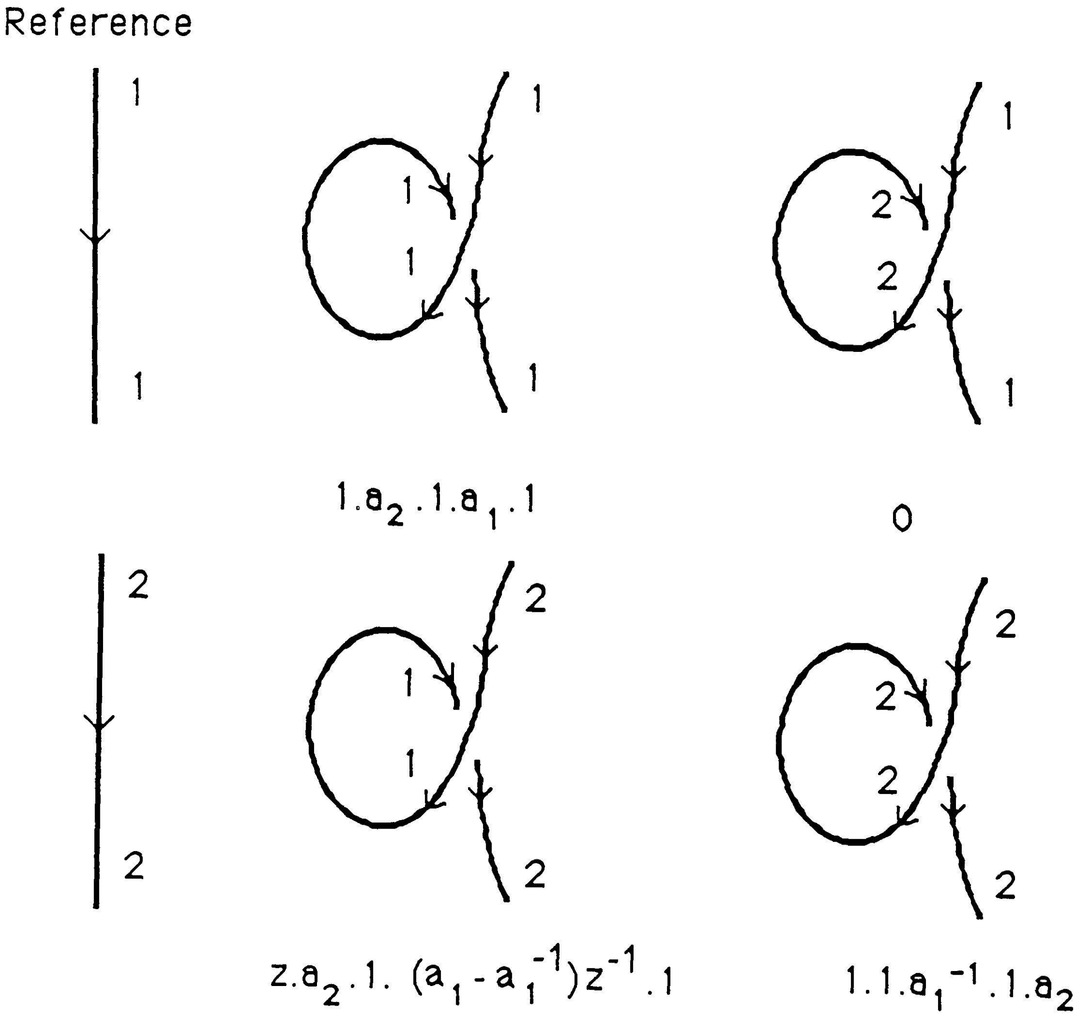

FIGURE 10

The proof is described on Figures 9 (move of type A'_1) and 10 (move of type A'_3) with the same conventions as above. In each case properties (ii) and (v) of H' are used.

Proof of property (i). First we show on Figure 11 that the move of type B_2 can be reduced, by using the already established property (iii), to the move of type B_1. As before, in this figure each diagram D stands for $H''(D, z, a_1, a_2)$.

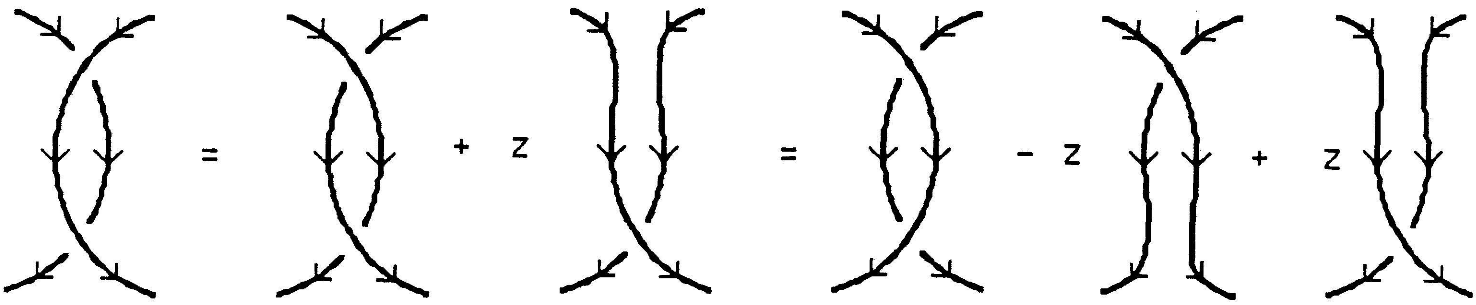

FIGURE 11

Let D' be obtained from D by a move of type B'_1, B'_3 or B'_4 (see Figure 2B), whose effect is studied in Figures 12, 13, 14 respectively. Let f be a given outer labelling. For each move there are six cases to consider according to the labels of the boundary edges. We take as a reference $R = D$ and the unique element h of $L(D, 2)$ such that $f \subseteq h$ whenever such an h exists (this corresponds to the first four cases). Then, using property (i) for H' in the two first cases, it is easy to check directly on Figures 12, 13, 14 that $C'(D', f) = 1$.

In the remaining two cases there is no labelling g in $L(D, 2)$ such that $f \subseteq g$. We choose a suitable reference and, using property (ii) for H' in Figures 13 and 14, we check that $C'(D', f) = 0$.

Finally let D' be obtained from D by a move of type C (see Figure 2C), and consider an outer labelling f. We first classify the different cases for f according to the labellings of the boundary edges which are oriented from the exterior of the separator towards its interior. For each case we describe for both diagrams D and D' all labellings of the edges incident to the inner vertices which will yield a non-zero inner interaction. This is done in Figures 15 to 21. In each figure one part corresponds to D and the other to D', and each labelling appears with its inner interaction.

We must show that, after the choice of a suitable reference, for every outer labelling f, $C'(D, f) = C'(D', f)$.

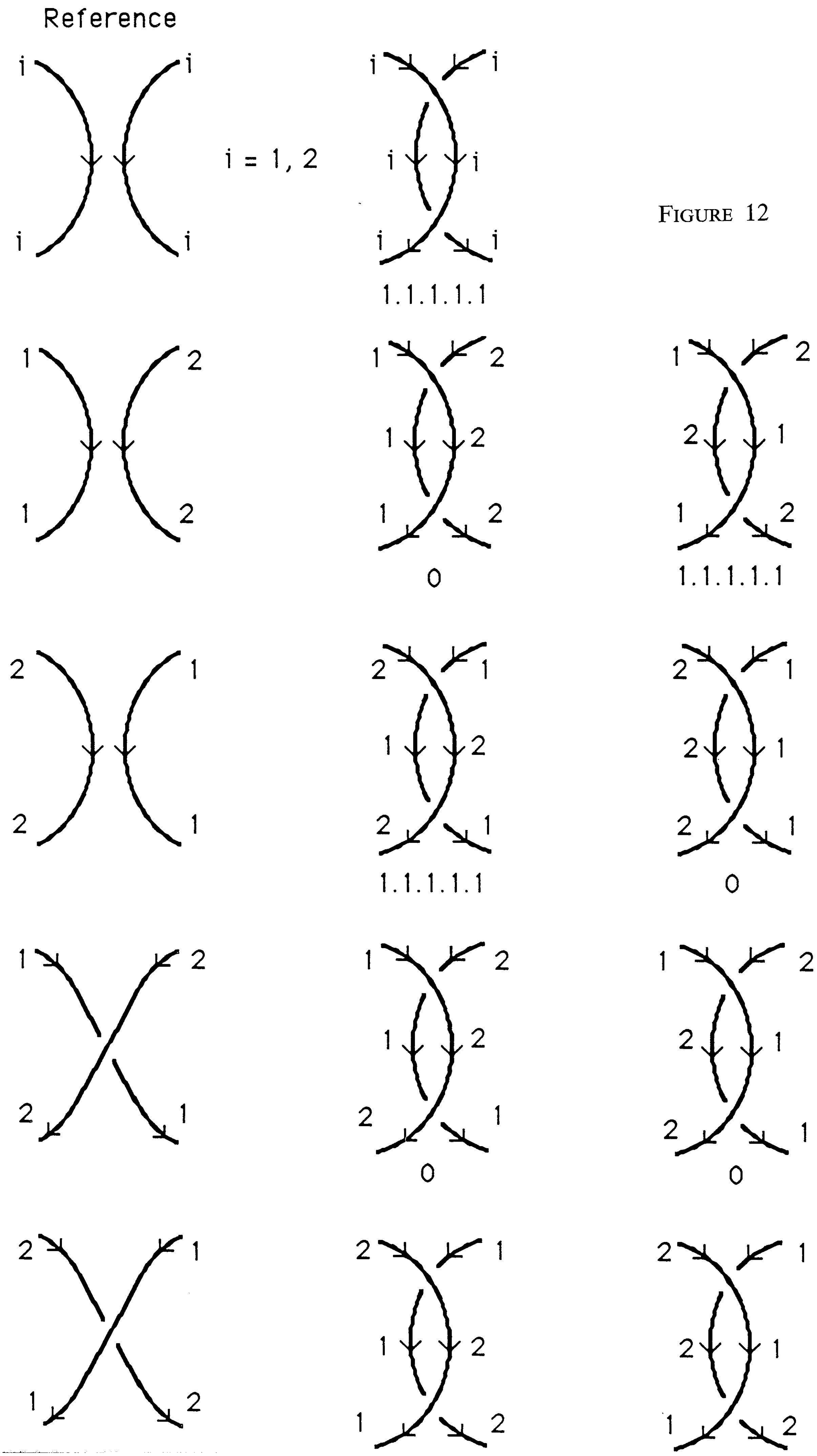

FIGURE 12

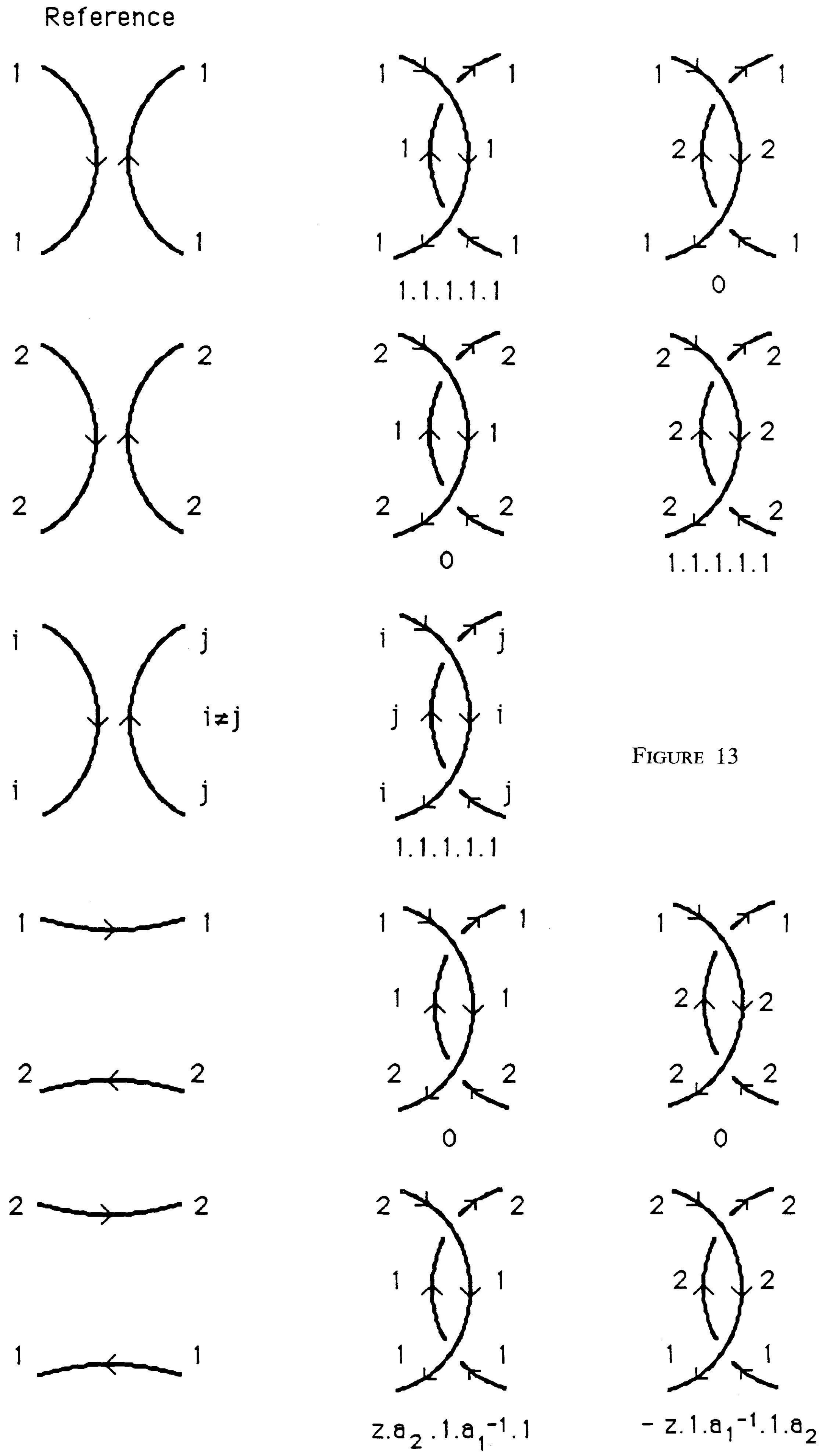

FIGURE 13

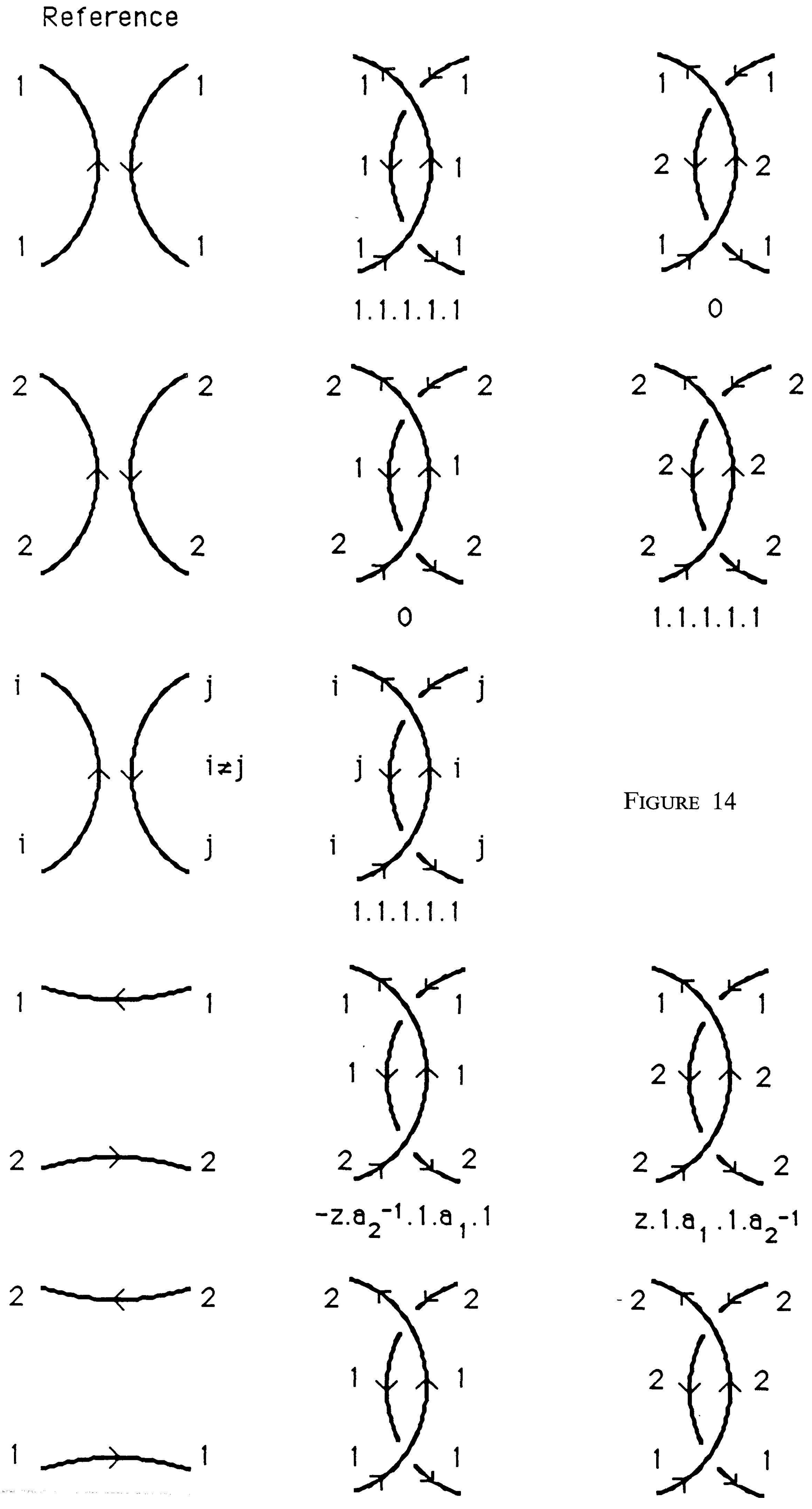

FIGURE 14

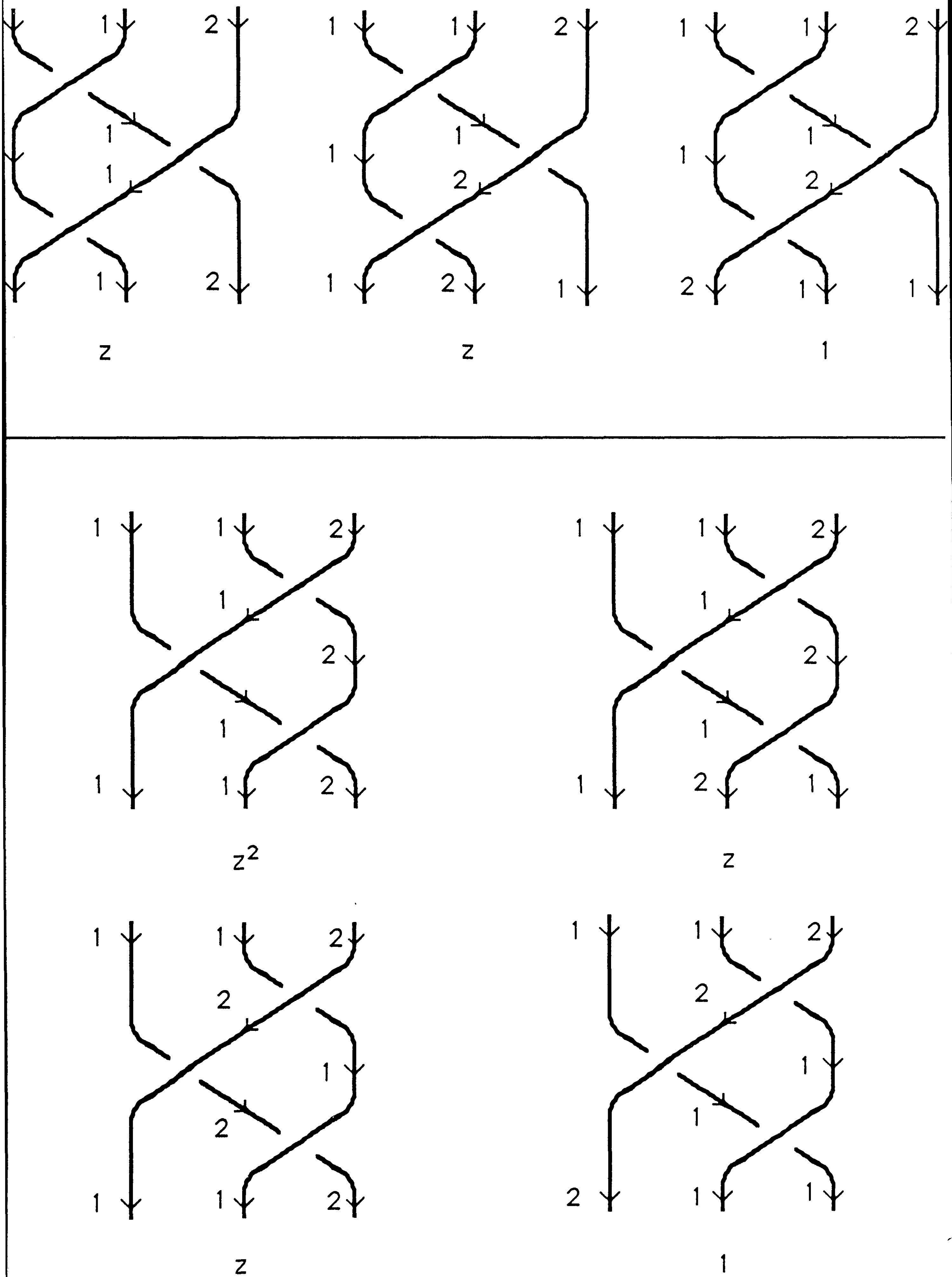

FIGURE 15

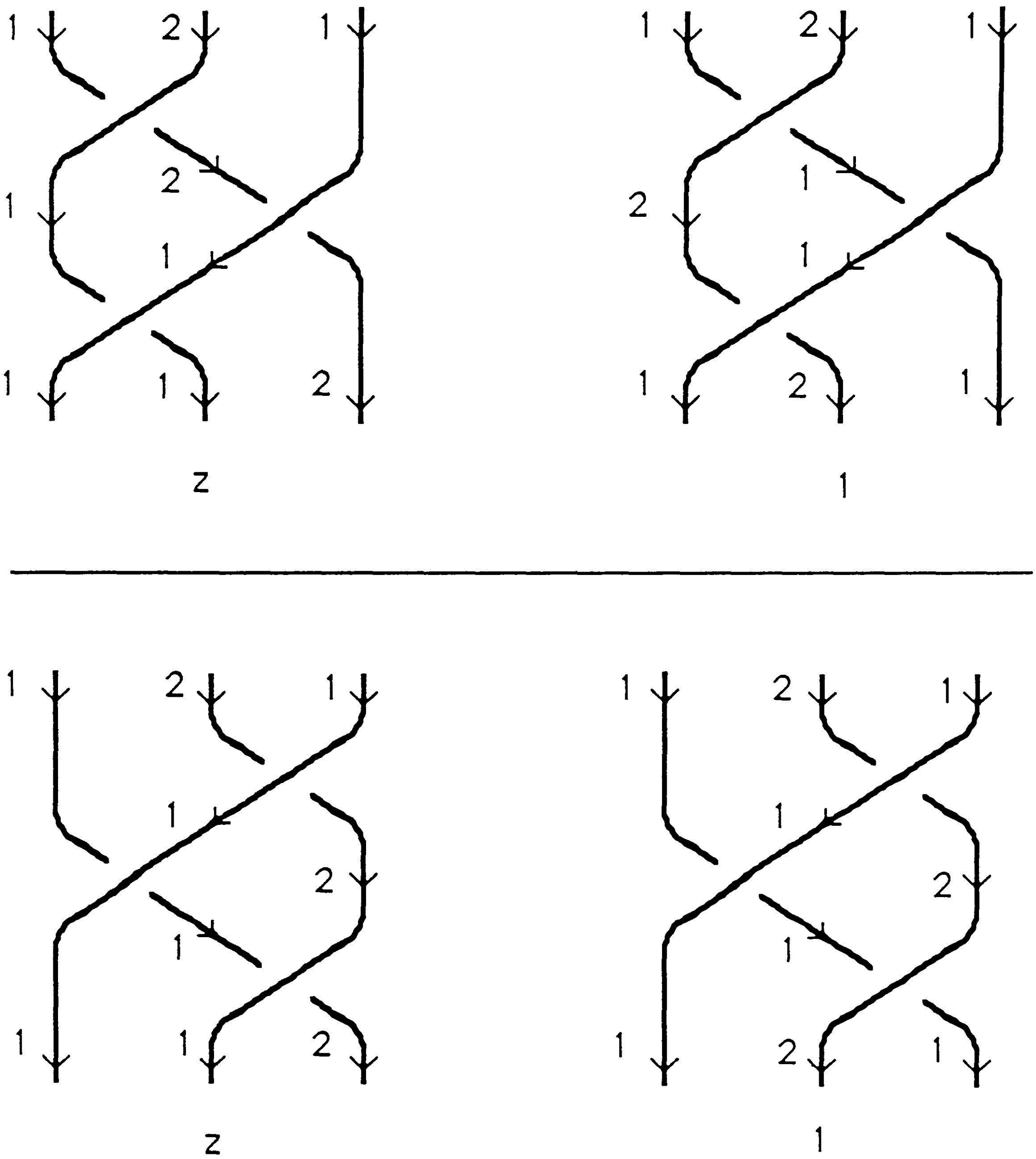

FIGURE 16

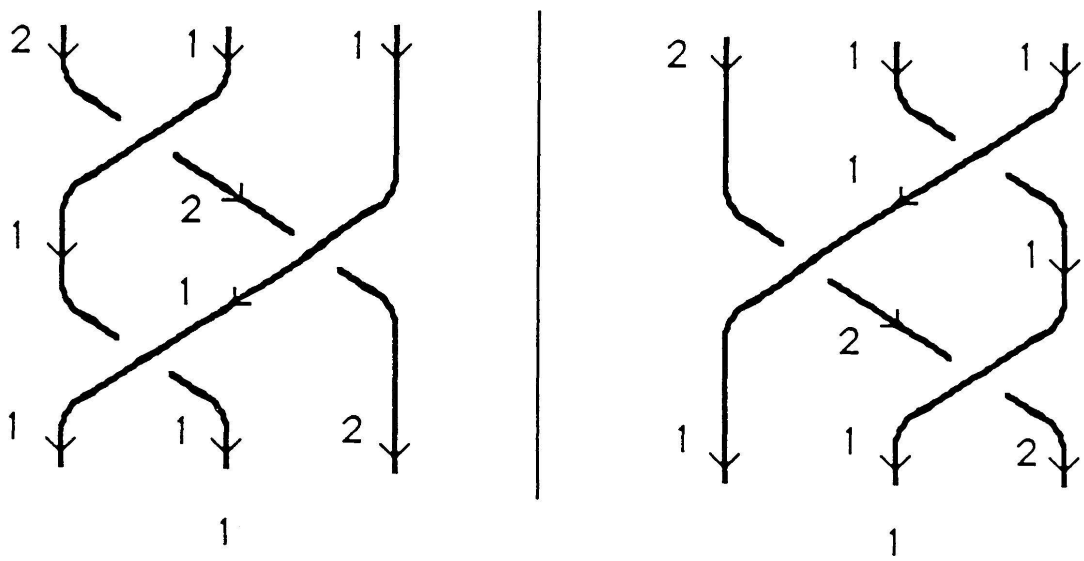

FIGURE 17

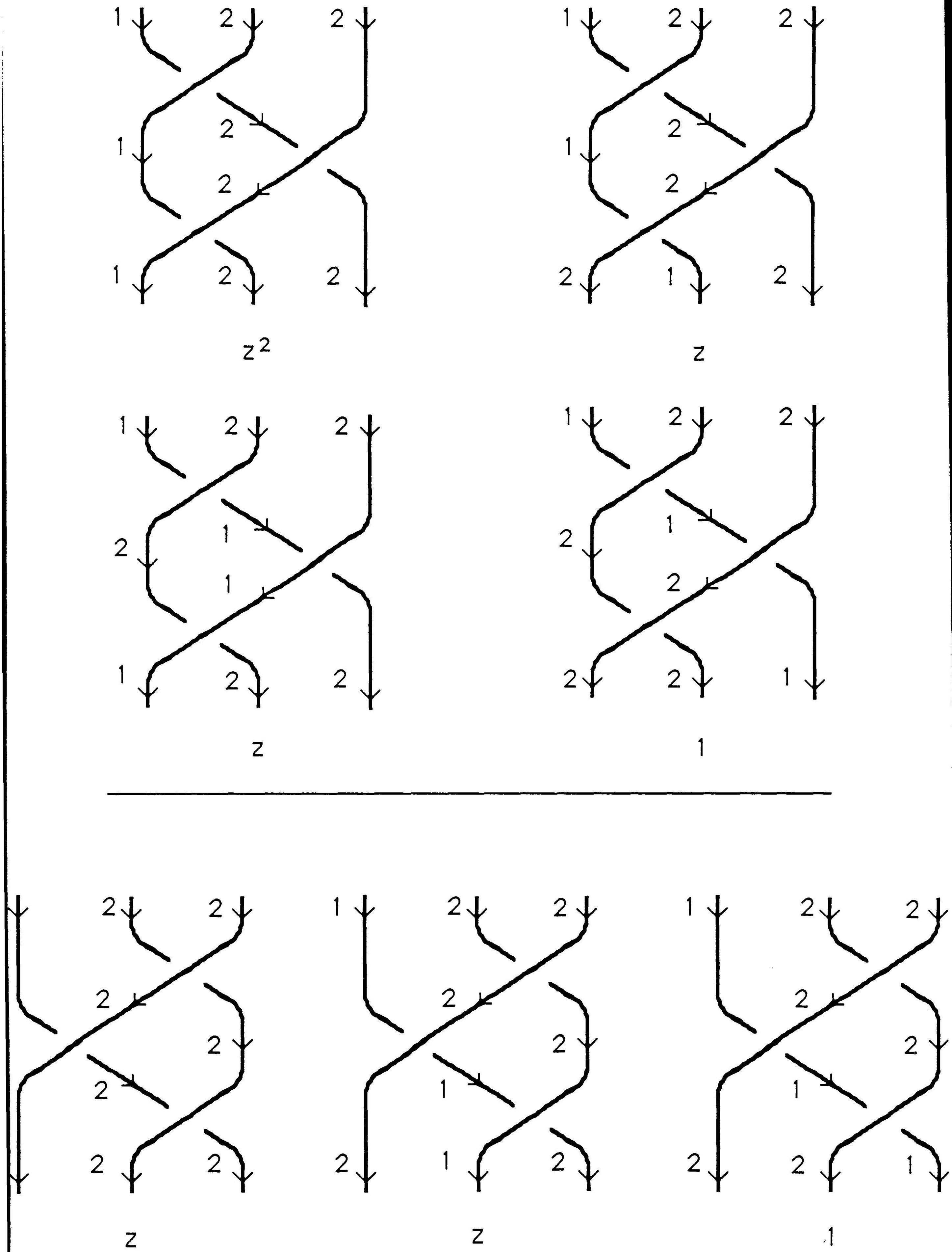

FIGURE 18

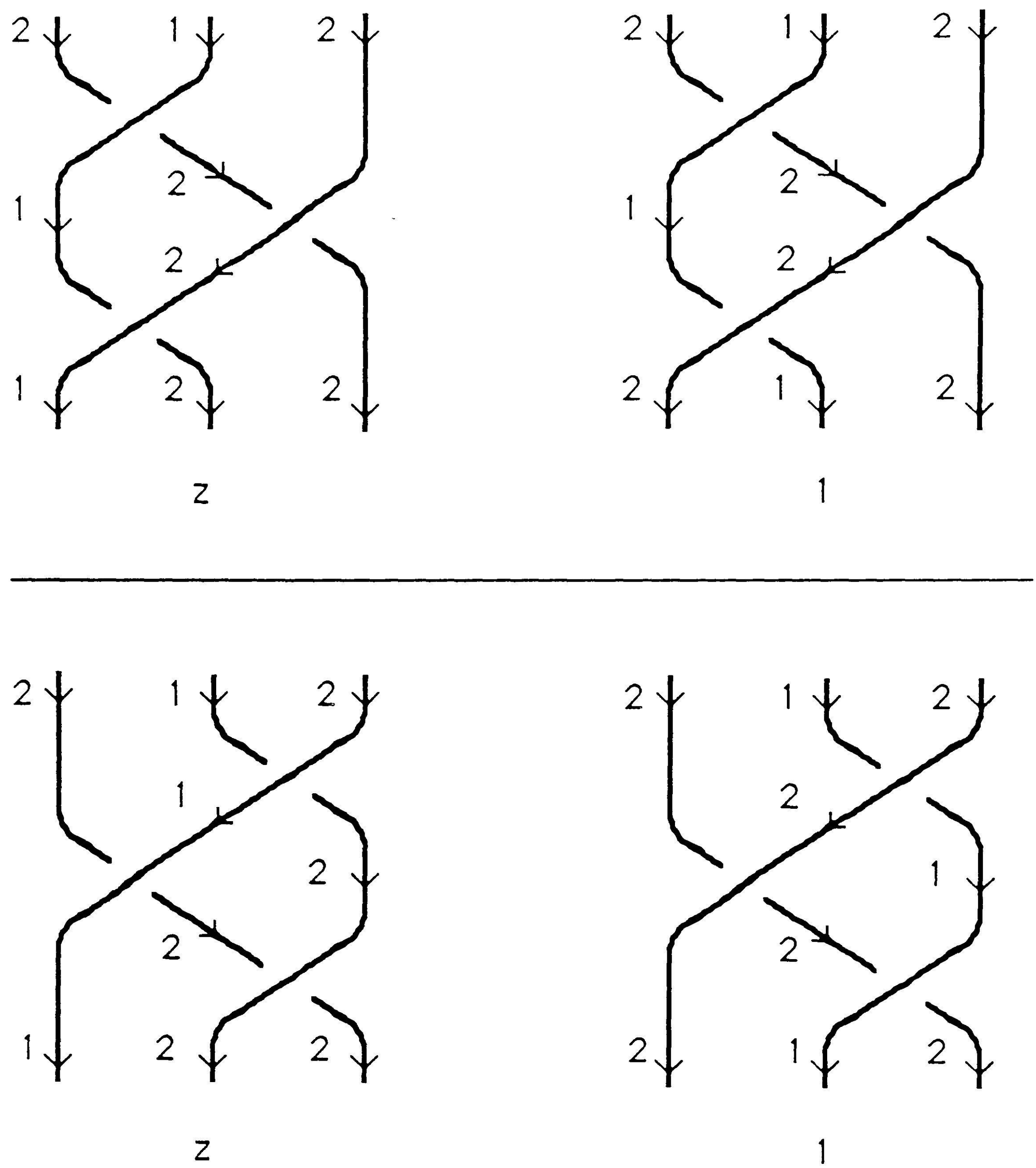

FIGURE 19

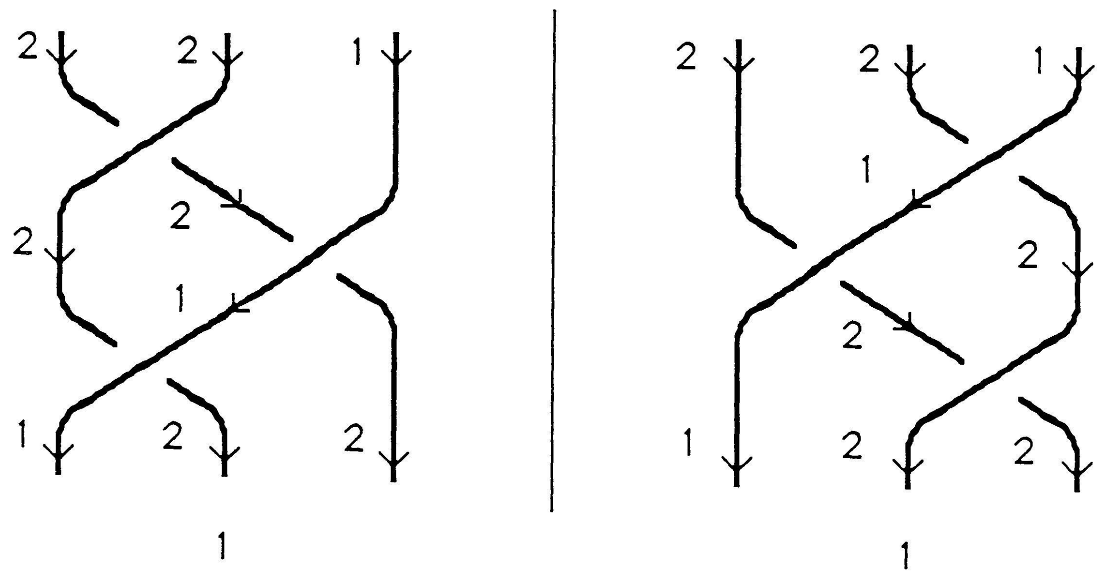

FIGURE 20

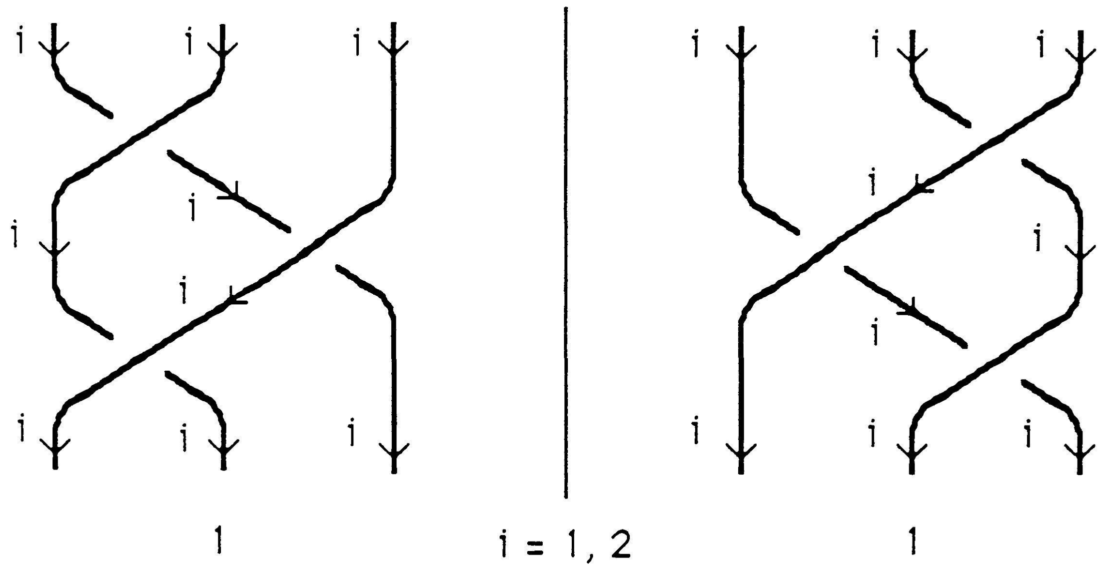

FIGURE 21

Clearly we may restrict our attention to the elements of $L(D, 2)$ and $L(D', 2)$ whose behavior inside the separator is depicted in one of the Figures 15 to 21. Thus it remains to perform the following analysis for each one of these figures: divide it into "subfigures" according to the labellings of the boundary edges which are oriented from the interior of the separator towards its exterior. Each subfigure will correspond to a certain class of outer labellings characterized by their value on the boundary edges. Then for each subfigure choose an appropriate reference and check that $C'(D, f) = C'(D', f)$. This is immediate in Figures 16, 17, 19, 20 (subdiagrams of D and D' are in bijective correspondence) and in Figure 21 (by property (i) for H'). In Figures 15 and 18 we must use the property of H' described in Figure 22, which is an immediate consequence of properties (i) and (iii).

This completes the proof.

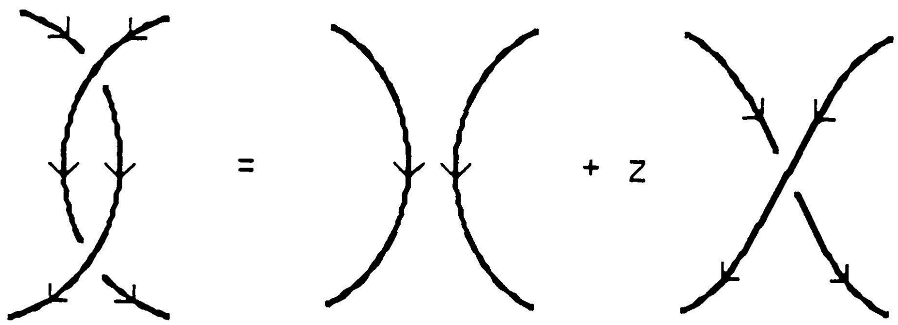

FIGURE 22

Remark. Figure 10 can be obtained from Figure 9 by a symmetry with respect to the vertical axis (without changing the crossing signs) together with the exchange of the numbers 1 and 2 in the labellings and the associated contributions. A similar relationship exists between Figures 13 and 14, 15 and 18, 16 and 19, 17 and 20 respectively. We could have used this to reduce the amount of case-checking in the proof of Proposition 1. However we found it simpler and more convincing to give the full set of figures.

2.3. A SEQUENCE OF STATE MODELS

We now derive from Proposition 1 state models for an infinite sequence of specializations of the homfly polynomial. This result appears in [20] and [28] where the original idea is attributed to [14].

We begin with a useful lemma.

Consider a labelling f of the diagram D together with a labelling g_i of $D_{f,i}$ for each i such that $f^{-1}(i)$ is not empty. We shall say that the labelling h of D is *compatible* with f and the g_i if for any two edges e, e' of D: if $f(e) < f(e')$ then $h(e) < h(e')$; if $f(e) = f(e') = i$ then $h(e) \leqslant h(e')$ if and only if $g_i(P_f(e)) \leqslant g_i(P_f(e'))$, where P_f is the projection associated to f (see Section 2.2).

UNIFICATION LEMMA. *For any labelling h of D compatible with f and the g_i,*

$$<D \mid h> \; = \; <D \mid f> \prod_i <D_{f,i} \mid g_i> .$$

This equality is easily proved by studying the possible contributions of a given vertex v to both sides. If v is incident only to edges labelled i by f, $<v \mid D \mid f> \; = 1$ and v is a vertex of $D_{f,j}$ if and only if $j = i$. Then the contribution of v to the right-hand side is $<v \mid D_{f,i} \mid g_i>$, which is clearly equal to $<v \mid D \mid h>$. If v is incident to edges labelled in two distinct ways by f, then v does not contribute to $\prod_i <D_{f,i} \mid g_i>$ and it is easy to check that $<v \mid D \mid h> \; = \; <v \mid D \mid f>$.

In the sequel we write $z = t - t^{-1}$.

PROPOSITION 2. *For any diagram D and positive integer q,*

$$H'(D, z, t^q) = t^{-(q+1)r(D)} \sum_{f \in L(D,q)} <D \mid f> t^{w(D,f) + 2s(D,f)},$$

where $w(D, f) = \sum_{i=1,\dots q} w(D_{f,i})$ *and* $s(D, f) = \sum_{i=1,\dots q} ir(D_{f,i})$.

Proof. We proceed by induction on q.

For $q = 1$, $L(D, q)$ contains only one element f for which $<D \mid f> = 1$, $w(D, f) = w(D)$ and $s(D, f) = r(D)$. The result reduces to: $H'(D, z, t) = t^{w(D)}$. This is easy to check and well known.

Assume now that the result holds for the positive integer q. By Proposition 1

$$H'(D, z, t^{q+1}) = \sum_{f \in L(D,2)} C(D, f),$$

with

$$C(D, f) = <D \mid f> t^{-r(D_{f,1})} t^{qr(D_{f,2})} H'(D_{f,1}, z, t^q) H'(D_{f,2}, z, t).$$

Let us fix f and write $D1$ for $D_{f,1}$, $D2$ for $D_{f,2}$.
By the induction hypothesis

$$H'(D1, z, t^q) = t^{-(q+1)r(D1)} \sum_{g \in L(D1,q)} <D1 \mid g> t^{w(D1,g)+2s(D1,g)}$$

and we have seen that $H'(D2, z, t) = t^{w(D2)}$.
It follows that $C(D, f)$ is equal to

$$<D \mid f> t^{-r(D1)+qr(D2)} t^{-(q+1)r(D1)} \sum_{g \in L(D1,q)} <D1 \mid g> t^{w(D1,g)+2s(D1,g)} t^{w(D2)}$$

Since $r(D1) + r(D2) = r(D)$, $C(D, f)$ can be rewritten as

$$t^{-(q+2)r(D)} \sum_{g \in L(D1,q)} <D \mid f> <D1 \mid g> t^{w(D1,g)+w(D2)+2s(D1,g)+(2q+2)r(D2)}.$$

Now for every labelling g in $L(D1, q)$ define a labelling h of D as follows. For an edge e of D, if $f(e) = 1$ then $h(e) = g(P_f(e))$; if $f(e) = 2$ then $h(e) = q + 1$. The labelling h clearly belongs to $L(D, q+1)$ and we shall denote it by $u(f, g)$.

We first note that

$$w(D1, g) + w(D2) = w(D, u(f, g))$$

and

$$s(D1, g) + (q+1) r(D2) = s(D, u(f, g)).$$

Moreover $u(f, g)$ is compatible with f, the labelling g of $D1$ and the constant labelling of $D2$ with value $q + 1$. Hence, by the Unification Lemma, $<D \mid f> <D1 \mid g> = <D \mid u(f, g)>$.

Using the above remarks, $C(D, f)$ can be rewritten as

$$t^{-(q+2)r(D)} \sum_{g \in L(D1,q)} <D \mid u(f, g)> t^{w(D,u(f,g))+2s(D,u(f,g))}.$$

Since u is easily seen to define a bijection from

$$\{(f, g)/f \in L(D, 2), g \in L(D_{f,1}, q)\}$$

to $L(D, q+1)$ we obtain

$$H'(D, z, t^{q+1}) = t^{-(q+2)r(D)} \sum_{h \in L(D, q+1)} <D \mid h> t^{w(D,h)+2s(D,h)}$$

as required.

Remarks. (1) The case $q = 2$ in Proposition 2 yields a state model for the Jones polynomial which, as noted in [28], can be directly related to Kauffman's "bracket polynomial" model [17, 18] via the theory of "ice-type models" developed in [2], Section 12.3.

(2) The state models of Proposition 2 can be used as shown in [28] to obtain a proof of the existence of the homfly polynomial.

(3) The applicability of Proposition 1 is limited by the fact that it cannot deal efficiently with the Alexander-Conway polynomial. This is because for every non-empty diagram D, $H'(D, z, 1) = 0$. Another aspect of this phenomenon is that property (v) cannot be defined in a coherent way for $H(D, z, 1) = A(D, z)$: the effect of adding a free loop to a non-empty diagram is qualitatively different from the corresponding effect on the empty diagram. However, a coherent version of property (v) is essential to the proof of Proposition 1. This has lead us to look for another form of the composition product which will be capable of handling the Alexander-Conway polynomial. So far we have been able to define such a composition product only in the case of closed braids. This is presented in the next section.

3. The specified composition product for closed braids

3.1. Braid words, braid diagrams and the specified product

Let us consider an infinite sequence of symbols (s_i) indexed by the set of positive integers. Artin's braid group on n strings $B_n (n \geqslant 1)$ can be defined by the presentation:

$$<s_1, \dots s_{n-1} \mid s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, i = 1, \dots n-2; s_i s_j = s_j s_i, |i-j| \geqslant 2> .$$

Thus B_1 is the trivial group and B_n is the subgroup of B_{n+1} generated by $s_1, \dots s_{n-1}$. We call *braid word on n strings* any word on the alphabet $\{s_i, s_i^{-1}/i = 1, \dots n-1\}$. Thus a braid word on n strings is also a braid word on n' strings for all $n' \geqslant n$. To every braid word m on n strings we

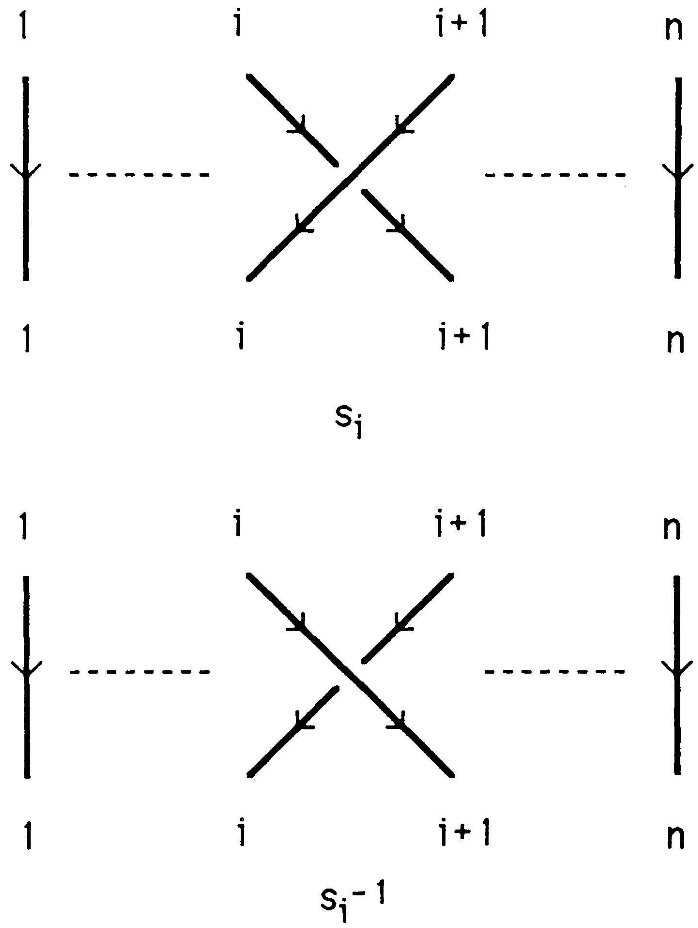

FIGURE 23

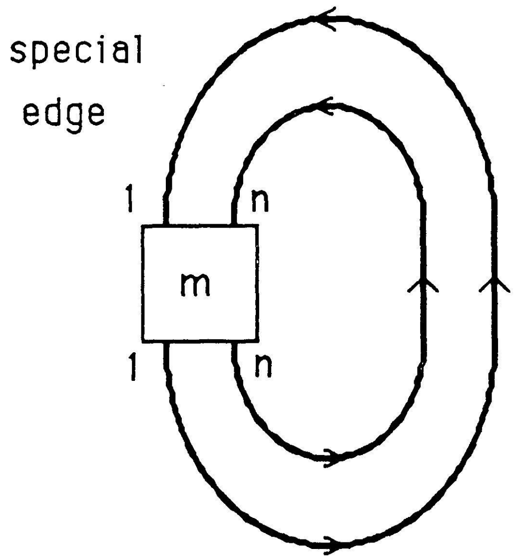

FIGURE 24

associate a *braid diagram on* n *strings* as follows. To each letter of m corresponds a portion of diagram, or *block*, according to the rule described on Figure 23. Each block has n top incoming *half-edges*, numbered from 1 to n in the left-to-right order, and n bottom outgoing half-edges which are numbered similarly. First the blocks are concatenated from top to bottom in the order of occurrence of the corresponding letters in m. Here the concatenation of two blocks corresponds to the merging of the bottom half-edges of the first block to the top half-edges with corresponding numbers of the second block. Finally the bottom half-edges are merged as shown on Figure 24 to the corresponding top half-edges, thus forming n *return edges*. The return edges will be numbered from 1 to n as the corresponding half-edges. The return edge which is numbered 1 will be called the *special edge*. We note that a braid diagram is non empty, and that the unique braid diagram on one string is a free loop. We also observe that any braid diagram on n strings has rotation number n.

A *specified labelling* of the braid diagram D is a labelling such that the special edge receives the label 1. We denote by $SL(D, k)$ the set of specified labellings of D which take their values in $\{1, \dots k\}$.

PROPOSITION 3. *For any braid diagram* D,

$$\sum_{f \in SL(D,2)} \langle D \mid f \rangle \, a_2^{-r(D_{f,1})+1} \, a_1^{r(D_{f,2})} \, H(D_{f,1}, z, a_1) \, H'(D_{f,2}, z, a_2) = H(D, z, a_1 a_2)\,.$$

Proof. Let $H''(D, z, a_1, a_2)$ denote the expression

$$\sum_{f \in SL(D,2)} \langle D \mid f \rangle \, a_2^{-r(D_{f,1})+1} \, a_1^{r(D_{f,2})} \, H(D_{f,1}, z, a_1) \, H'(D_{f,2}, z, a_2)$$

which is an element of the ring K of Laurent polynomials with integer coefficients in the variables z, a_1, a_2.

Let us introduce the following modifications in each step of the proof of Proposition 1: we replace $H'(D_{f,1}, z, a_1)$ by $H(D_{f,1}, z, a_1)$, we restrict our attention to specified labellings, and we assume that the special edge is outer. Then it is easy to check that, since $D_{f,1}$ is non-empty for every specified labelling f, our arguments (and especially those involving property (v) for $H(D_{f,1}, z, a_1)$) remain correct.

Thus in particular we have the following properties for braid diagrams on n strings D, D', D^+, D^-, D°:

(1) If D' is obtained from D by a move of type B_1, B_2 or C for which the return edges are outer (such a move will be called *inner*), then $H''(D', z, a_1, a_2) = H''(D, z, a_1, a_2)$.

(2) If (D^+, D^-, D°) form a Conway triple then:

$$H''(D^+, z, a_1, a_2) - H''(D^-, z, a_1, a_2) = zH''(D^\circ, z, a_1, a_2).$$

Clearly two braid words on n strings represent the same element of the group B_n if and only if the associated braid diagrams on n strings can be obtained one from the other by a finite sequence of inner moves of types B_1, B'_1, B_2, B'_2 or C, C' (the insertion or deletion of trivial relators $s_i s_i^{-1}$ or $s_i^{-1} s_i$ corresponds to the moves of types B_1, B'_1, B_2, B'_2; we associate in a similar way the relations $s_i s_j = s_j s_i$, $|i-j| \geqslant 2$ to plane deformations and the relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ to the moves of types C, C').

Hence it follows from (1) that we may define a mapping H''_n from B_n to K as follows. For every element b of B_n, $H''_n(b) = H''(D, z, a_1, a_2)$ for any braid diagram on n strings D associated to a braid word which represents b.

Consider now the quotient A_n of the group algebra $K(B_n)$ by the (two-sided) ideal J generated by the elements $s_i - s_i^{-1} - z\,(i=1, ..., n-1)$. Let us extend H''_n to $K(B_n)$ by linearity. Then it follows from (2) that H''_n takes the same value on two elements of $K(B_n)$ which are congruent modulo J. We shall now identify H''_n with the induced linear functional from A_n to K. The embedding of B_n into B_{n+1} is extended to an embedding of A_n into A_{n+1} in the obvious way. Thus H''_n is well defined on $A_1, \dots A_n$.

Coming back to our modified version of the proof of Proposition 1, we obtain the following properties for all $n \geqslant 1$.

(3) If $u \in A_n$ then $H''_{n+1}(u) = dH''_n(u)$ with $d = \left(a_1 a_2 - (a_1 a_2)^{-1}\right) z^{-1}$.

(4) If $u, v \in A_n$ then $H''_{n+1}(us_n v) = aH''_n(uv)$ and $H''_{n+1}(us_n^{-1} v) = a^{-1} H''_n(uv)$ with $a = a_1 a_2$.

Indeed it is enough to prove (3) and (4) for u, v in B_n. Then (3) corresponds to the addition of a single free loop to a non-empty diagram, and (4) to a move of type A'_1 or A'_2.

It is also easily checked that when D is the unique braid diagram on one string, that is, the free loop with rotation number 1, $H''(D, z, a_1, a_2) = 1$. Since B_1 is trivial, A_1 can be identified with K. Hence

(5) H''_1 is the identity mapping on K.

We now show that

(6) If $u, v \in A_n$ then $H''_n(uv) = H''_n(vu)$.

The following proof is essentially the same as those given in [7] and [13] for Ocneanu's trace [25]. However since the context is slightly different we feel necessary to give the details. We proceed by induction on n. The result is trivial for $n = 1$ by (5). Assume that the result holds for A_n. It is enough to show that for all u in A_{n+1} and i in $1, \ldots, n$, $H''_{n+1}(us_i) = H''_{n+1}(s_i u)$. We need the following basic result: A_{n+1} is generated as a K-algebra by $G_{n+1} = A_n \cup \{xs_n y / x, y \in A_n\}$ (see [7], [13], [31]). Now we may assume that u belongs to G_{n+1}.

If $u \in A_n$ and $i < n$, by our induction hypothesis $H''_n(us_i) = H''_n(s_i u)$ and property (3) yields the result. Similarly by property (4), for u in A_n, $H''_{n+1}(us_n) = aH''_n(u) = H''_{n+1}(s_n u)$.

Assume now that $u = xs_n y$ with $x, y \in A_n$. If $i < n$, $H''_{n+1}(xs_n ys_i) = aH''_n(xys_i)$ and $H''_{n+1}(s_i xs_n y) = aH''_n(s_i xy)$; the result then follows from the induction hypothesis. It remains to prove that $H''_{n+1}(xs_n ys_n) = H''_{n+1}(s_n xs_n y)$. We may assume that $x, y \in G_n$. There are four cases to consider.

— If x and y both belong to A_{n-1} they commute with s_n and the result is immediate.

— If $x = x's_{n-1}x''$ with $x', x'' \in A_{n-1}$:

$$s_n xs_n y = s_n x's_{n-1}x''s_n y = x's_n s_{n-1}s_n x''y = x's_{n-1}s_n s_{n-1}x''y .$$

Then by property (4)

$$\begin{aligned} H''_{n+1}(s_n xs_n y) &= aH''_n(x's^2_{n-1}x''y) = aH''_n(x'(zs_{n-1}+1)x''y) \\ &= azH''_n(xy) + aH''_n(x'x''y) . \end{aligned}$$

Now if $y \in A_{n-1}$

$$xs_n ys_n = xys_n^2 = xy(zs_n + 1) .$$

Then by properties (3) and (4)

$$H''_{n+1}(xs_n ys_n) = zaH''_n(xy) + dH''_n(xy) .$$

The result follows since $dH''_n(x's_{n-1}x''y) = aH''_n(x'x''y) = adH''_{n-1}(x'x''y)$.

On the other hand if $y = y's_{n-1}y''$ with $y', y'' \in A_{n-1}$

$$xs_n ys_n = xs_n y's_{n-1}y''s_n = xy's_n s_{n-1}s_n y'' = xy's_{n-1}s_n s_{n-1}y''$$

and hence

$$\begin{aligned} H''_{n+1}(xs_n ys_n) &= aH''_n(xy's^2_{n-1}y'') = aH''_n(xy'(zs_{n-1}+1)y'') \\ &= azH''_n(xy) + aH''_n(xy'y'') . \end{aligned}$$

The result follows since

$$H''_n(x'x''y) = H''_n(x'x''y's_{n-1}y'') = aH''_{n-1}(x'x''y'y'') = H''_n(x's_{n-1}x''y'y'')$$
$$= H''_n(xy'y'').$$

— In the only remaining case $x \in A_{n-1}$ and $y = y's_{n-1}y''$ with $y', y'' \in A_{n-1}$. We have just seen that $H''_{n+1}(xs_nys_n) = azH''_n(xy) + aH''_n(xy'y'')$. We may also write

$$s_nxs_ny = s_n^2xy = (zs_n+1)xy,$$

so that $H''_{n+1}(s_nxs_ny) = zaH''_n(xy) + dH''_n(xy)$.

The result follows since $dH''_n(xy's_{n-1}y'') = aH''_n(xy'y'') = adH''_{n-1}(xy'y'')$.

Now we can use the construction of the homfly polynomial described in [7], [13]. Let us consider two braid diagrams D_1 on n_1 strings and D_2 on n_2 strings. For $i = 1, 2$ let b_i be the element of the braid group on n_i strings associated to D_i. Markov's theorem asserts that D_1 and D_2 are isotopic if and only if b_1 can be obtained from b_2 by a finite sequence of moves of one of the following types:

Markov move of type 1: replace $b \in B_n$ by a conjugate $cbc^{-1} (c \in B_n)$.

Markov move of type 2: replace $b \in B_n$ by $bs_n \in B_{n+1}$ or $bs_n^{-1} \in B_{n+1}$, or perform the converse operation.

It then follows from properties (4) and (6) that if D_1 and D_2 are isotopic:

$$(a_1a_2)^{-w(D_1)} H''(D_1, z, a_1, a_2) = (a_1a_2)^{-w(D_2)} H''(D_2, z, a_1, a_2).$$

Since by the classical result of Alexander every diagram is isotopic to some braid diagram, there exists an isotopy invariant $P''(D, z, a_1, a_2)$ defined for all diagrams D which is equal to $(a_1a_2)^{-w(D)} H''(D, z, a_1, a_2)$ for every braid diagram D. Using property (2) and a refinement of Alexander's Theorem one easily shows as in [7] p. 294 or [13] p. 348 that this invariant satisfies the equation

$$a_1a_2 P''(D^+, z, a_1, a_2) - (a_1a_2)^{-1} P''(D^-, z, a_1, a_2) = zP''(D^\circ, z, a_1, a_2)$$

for every Conway triple (D^+, D^-, D°). Then property (5) allows us to identify $P''(D, z, a_1, a_2)$ with the homfly polynomial $P(D, z, a_1a_2)$, and thus to complete the proof.

Remark. In Proposition 3 we may replace $r(D_{f,i})$ by the number of return edges labelled i by f.

3.2. A SIMPLIFIED STATE MODEL

We give here a "specified" version of Proposition 2. As before we write $z = t - t^{-1}$.

PROPOSITION 4. *For any braid diagram* D *and positive integer* q,

$$H(D, z, t^q) = t^{q-1-(q+1)r(D)} \sum\nolimits_{f \in SL(D, q)} \langle D \mid f \rangle \, t^{w(D, f) + 2s(D, f)} ,$$

where $w(D, f) = \sum_{i=1,\dots q} w(D_{f,i})$ *and* $s(D, f) = \sum_{i=1,\dots q} ir(D_{f,i})$.

The proof is almost identical to that of Proposition 2 and will be omitted. Note that $H(D, z, t^q)$ is now expressed as a Laurent polynomial in t (this was not the case in Proposition 2 which only gave such an expression for $H'(D, z, t^q)$).

3.3. MODELS FOR THE ALEXANDER-CONWAY POLYNOMIAL

We begin with the following identity which is immediately obtained by setting $a_1 = a$ and $a_2 = a^{-1}$ in Proposition 3:

PROPOSITION 5. *For any braid diagram* D,

$$A(D, z) = a^{r(D)-1} \sum\nolimits_{f \in SL(D, 2)} \langle D \mid f \rangle \, H(D_{f,1}, z, a) \, H'(D_{f,2}, z, a^{-1}) .$$

Remark: For a diagram D, let $D^{\sim}$ denote the *mirror image* of D, that is, the diagram obtained from D by changing the signs of all vertices. It is easy to deduce from the Theorem of section 2.1 that $H(D^{\sim}, z, a) = H(D, -z, a^{-1}) = (-1)^{c(D)-1} H(D, z, a^{-1})$, where $c(D)$ denotes the number of components of D (in the sense of knot theory, not graph theory). These identities can be used to reformulate Proposition 5.

We now recall that for $z = t - t^{-1}, H(D, z, t) = t^{w(D)}$. Similarly, $H(D, z, -t^{-1}) = (-t)^{-w(D)}$. It easily follows that

$$H'(D, z, t^{-1}) = (-1)^{r(D)} (-t)^{-w(D)} .$$

Taking $a = t$ in Proposition 5 we obtain:

PROPOSITION 6. *For any braid diagram* D,

$$A(D, z) = t^{r(D)-1} \sum\nolimits_{f \in SL(D, 2)} \langle D \mid f \rangle \, t^{w(D_{f,1})} (-t)^{-w(D_{f,2})} (-1)^{r(D_{f,2})} .$$

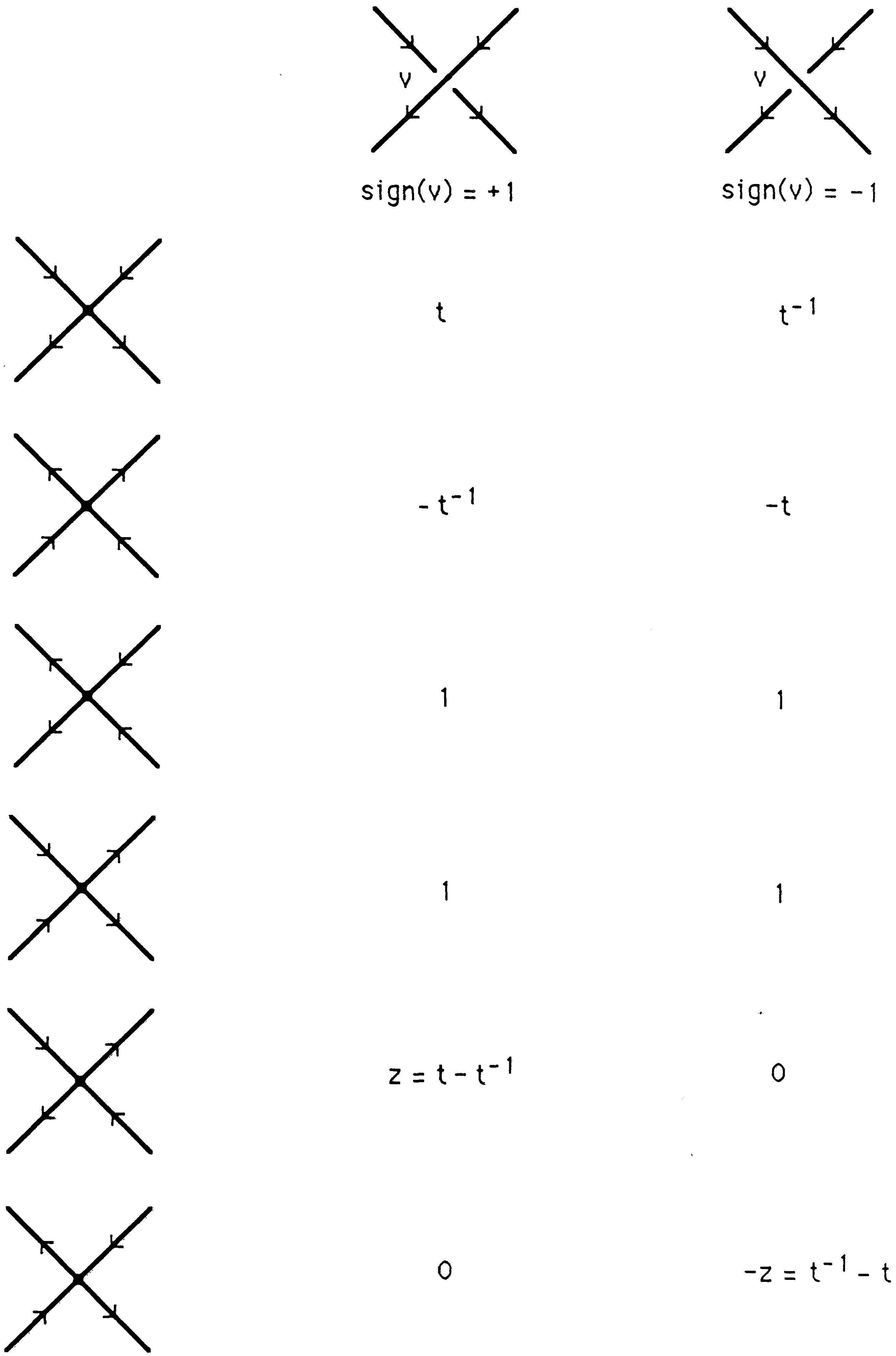

FIGURE 25

Remark. In the above expression for $A(D, z)$ we may replace t by $-t^{-1}$.

Proposition 6 yields an "ice-type" model (see [2]) for the Alexander-Conway polynomial. Let us call *Eulerian orientation* of a diagram D every rearrangement of the edge orientations such that at every vertex v the number of edges incident towards v equals the number of such edges incident from v (a loop at v contributing 1 to both numbers). We shall denote by $O(D)$ the set of Eulerian orientations of D. It is easy to see that for every labelling f in $L(D, 2)$ if we reverse the orientations of all edges labelled 2 we obtain an Eulerian orientation of D, and that this defines a bijective correspondence from $L(D, 2)$ to $O(D)$. Moreover when D is a braid diagram, if we denote by $SO(D)$ the set of Eulerian orientations of D such that the orientation of the special edge is not changed, we obtain a bijection from $SL(D, 2)$ to $SO(D)$. For a vertex v of D and an Eulerian orientation o of D let us define their *interaction* $<v \mid o>$ as on Figure 25. If D has vertex-set V let us write $<D \mid o> = \prod_{v \in V} <v \mid o>$. Finally let $r(o)$ denote the number of return edges of D which are reversed in the orientation o. Then we may reformulate Proposition 6 as follows.

PROPOSITION 7. *For any braid diagram D on n strings,*

$$A(D, z) = t^{n-1} \sum_{o \in SO(D)} (-1)^{r(o)} <D \mid o>$$

3.4. THE ALEXANDER EXPANSION FOR THE HOMFLY POLYNOMIAL

We begin with the following immediate consequence of Proposition 3.

PROPOSITION 8. *For any braid diagram D,*

$$H(D, z, a) = \sum_{f \in SL(D, 2)} <D \mid f> a^{-r(D_{f,1})+1} A(D_{f,1}, z) H'(D_{f,2}, z, a).$$

Let D be a braid diagram on n strings and denote its return edges by $e_1, \dots e_n$ in left to right order, e_1 being the special edge. For a labelling f of D, let us call *pattern* of f the partition of $\{e_1, \dots e_n\}$ defined by all the non-empty $f^{-1}(i)$. We shall call a labelling f of D *compressed* if the word $f(e_1) \dots f(e_n)$ is lexicographically minimal (for the usual ordering of the integers) in the class of all words of the form $f'(e_1) \dots f'(e_n)$ where f' is a labelling with the same pattern as f. We denote by $CL(D)$ the set of compressed labellings of D and by $k(f)$ the cardinality of the image of the labelling f.

PROPOSITION 9. *For any braid diagram D,*

$$H(D, z, a) = \sum_{f \in CL(D)} <D \mid f> a^{-r(D)+k(f)} \left((a-a^{-1})z^{-1}\right)^{k(f)-1} \prod_{i=1,\dots k(f)} A(D_{f,i}, z).$$

Proof. We proceed by induction on the rotation number of D. The result is trivial if this number is 1. Now by Proposition 8, $H(D, z, a) = \sum_{f \in SL(D,2)} C(D, f)$, with

$$C(D, f) = <D \mid f> a^{-r(D_{f,1})+1} A(D_{f,1}, z) H'(D_{f,2}, z, a).$$

Let us fix f and write $D2$ for $D_{f,2}$. If $D2$ is empty, $H'(D2, z, a) = 1$. Otherwise $D2$ is a braid diagram whose special edge is the image under the projection P_f associated to f of the leftmost return edge of D which is labelled 2 by f. Since $D_{f,1}$ is not empty, $r(D2) < r(D)$. Then by our induction hypothesis, $H'(D2, z, a) = (a-a^{-1})z^{-1}H(D2, z, a)$ is equal to $\sum_{g \in CL(D2)} <D2 \mid g> a^{-r(D2)+k(g)} \left((a-a^{-1})z^{-1}\right)^{k(g)} \prod_{i=1,\dots k(g)} A(D2_{g,i}, z)$. Thus if $D2$ is not empty, since $r(D_{f,1}) + r(D2) = r(D)$, $C(D, f)$ is equal to

$$\sum_{g \in CL(D2)} <D \mid f> <D2 \mid g> a^{-r(D)+k(g)+1} \left((a-a^{-1})z^{-1}\right)^{k(g)} A(D_{f,1}, z) \prod_{i=1,\dots k(g)} A(D2_{g,i}, z).$$

For every g in $CL(D2)$, define a labelling $h = u(f, g)$ of D as follows. For an edge e of D, if $f(e) = 1$ then $h(e) = 1$; if $f(e) = 2$ then $h(e) = g(P_f(e)) + 1$, where P_f is the projection associated to f. Since f is specified and g is compressed, h is easily seen to be compressed. Clearly $k(h) = k(g) + 1$, $D_{f,1} = D_{h,1}$ and for $i = 1, \dots k(g)$, $D2_{g,i} = D_{h,i+1}$. Moreover h is compatible with f, the constant labelling of $D1$ with value 1 and the labelling g of $D2$. Hence, by the Unification Lemma, $<D \mid f> <D2 \mid g> = <D \mid h>$.

It follows that, if $D2$ is not empty, $C(D, f)$ is equal to

$$\sum_{g \in CL(D2),\, h=u(f,g)} <D \mid h> a^{-r(D)+k(h)} \left((a-a^{-1})z^{-1}\right)^{k(h)-1} \prod_{i=1,\dots k(h)} A(D_{h,i}, z).$$

For any compressed labelling h of D define

$$C'(D, h) = <D \mid h> a^{-r(D)+k(h)} \left((a-a^{-1})z^{-1}\right)^{k(h)-1} \prod_{i=1,\dots k(h)} A(D_{h,i}, z).$$

Denoting by f_0 the labelling of D which takes the value 1 on each edge, it remains to prove:

$$\sum_{h \in CL(D)} C'(D, h) = C(D, f_0) + \sum_{f \in SL(D,2),\, k(f)=2} \sum_{g \in CL(D2),\, h=u(f,g)} C'(D, h).$$

It is easy to check that $C'(D, f_0) = C(D, f_0) = a^{-r(D)+1} A(D, z)$. Moreover u

defines a bijection from $\{(f, g)/f \in SL(D, 2), k(f) = 2, g \in CL(D_{f,2})\}$ to $CL(D) - \{f_0\}$. This completes the proof.

3.5. Some consequences

Let D be a braid diagram on n strings and consider its homfly polynomial $P(D, z, a) = a^{-w(D)} H(D, z, a)$. Let $E(D)$ (respectively: $e(D)$) be the maximum (respectively: minimum) degree in the variable a of the Laurent polynomial $P(D, z, a)$. The following result is due to Franks and Williams [4] and Morton [22]. In fact in [22] it is generalized to arbitrary diagrams (the number of Seifert circles replacing the number of strings).

Proposition 10. *For any braid diagram D on n strings,*

$$1 - n - w(D) \leqslant e(D) \leqslant E(D) \leqslant n - 1 - w(D).$$

Proof. Using Proposition 9 we may write

$$P(D, z, a) = \sum_{f \in CL(D)} Q(D, f, z)\, a^{-w(D)-n+1} (a^2 - 1)^{k(f)-1},$$

where $Q(D, f, z) = \langle D \mid f \rangle z^{-k(f)+1} \prod_{i=1,\dots k(f)} A(D_{f,i}, z)$ is a Laurent polynomial in z. The result follows immediately since $1 \leqslant k(f) \leqslant n$.

Remark. For fixed z and n the fact that $H(D, z, a)$ is up to a constant factor a polynomial of degree at most $n - 1$ in the variable $a^2 - 1$ can be used to express it as a linear combination of the form $\sum_{i=1,\dots n} \lambda_i H(D, z, a_i)$ for n mutually independent variables a_i. The coefficients λ_i are given explicitly by Murakami in [23].

The following result is the specialization to braid diagrams of another result of Morton [22]. Let us denote by $M(D)$ the maximum degree in the variable z of the Laurent polynomial $P(D, z, a)$ (or equivalently $H(D, z, a)$).

Proposition 11. *For any braid diagram D on n strings with vertex-set V,*

$$M(D) \leqslant |V| - n + 1.$$

Proof. Coming back to Proposition 6, we see that every term in the expansion of $A(D, z)$ in powers of t is, up to sign, of the form

$$t^{n-1}(t - t^{-1})^{h(D,f)}\, t^{w(D_{f,1}) - w(D_{f,2})}$$

for some f in $SL(D, 2)$, where $h(D, f)$ is the number of vertices where $D_{f,1}$ and $D_{f,2}$ are mutually tangent. The minimum degree of such a term is not less than $n - 1 - |V|$. Hence the maximum degree in z of $A(D, z)$ is not greater than $|V| - n + 1$ (this result is generalized to arbitrary diagrams in [16] p. 120).

Now applying this to all the $D_{f,i}$ in the expression

$$Q(D, f, z) = \langle D \mid f \rangle\, z^{-k(f)+1} \prod_{i=1,\dots k(f)} A(D_{f,i}, z)$$

and noting that the vertices of the $D_{f,i}$ contribute 1 to the product $\langle D \mid f \rangle$ we easily obtain the desired inequality.

3.6. A STATE MODEL FOR THE HOMFLY POLYNOMIAL

We may combine Propositions 6 and 9 to obtain a state model for the homfly polynomial.

Consider a compressed labelling f of the braid diagram D together with a specified labelling g_i in $SL(D_{f,i}, 2)$ for each $i = 1, \dots k(f)$, and define the labelling $h = u(f, g_1, \dots g_{k(f)})$ as follows. For every edge e of D, if $f(e) = i$ then $h(e) = 2(i-1) + g_i(P_f(e))$, where P_f is the projection associated to f. Any labelling which can be obtained in this way will be called *half-compressed* and we shall denote by $C'L(D)$ the set of half-compressed labellings of D. For a labelling f of D we denote by $w_o(D, f)$ (respectively: $w_e(D, f)$) the sum $\sum_{i=1,\dots k(f)} w(D_{f,i})$ restricted to the odd (respectively: even) values of i. Similarly we define $r_e(D, f)$ as the sum $\sum_{i=1,\dots k(f)} r(D_{f,i})$ restricted to the even values of i. We denote by $k'(f)$ the number of distinct odd values taken by f.

PROPOSITION 12. *For any braid diagram* D, $H(D, z, a)$ *equals*

$$\sum_{f\in C'L(D)} \langle D \mid f \rangle\, (t^{-1}a)^{-r(D)+k'(f)} \left((a-a^{-1})z^{-1}\right)^{k'(f)-1} t^{w_o(D,f)} (-t)^{-w_e(D,f)} (-1)^{r_e(D,f)}.$$

Proof. Let f be a fixed compressed labelling of D. For each $i = 1, \dots k(f)$ let us write Di for $D_{f,i}$. By Proposition 6,

$$A(Di, z) = t^{r(Di)-1} \sum_{gi\in SL(Di, 2)} \langle Di \mid gi \rangle\, t^{w(Di_{gi,1})} (-t)^{-w(Di_{gi,2})} (-1)^{r(Di_{gi,2})}.$$

Let $T(D, f)$ be the set of $k(f)$-tuples $g = (gi, i=1, \dots k(f))$ with $gi \in SL(Di, 2)$ for all i. Then $\langle D \mid f \rangle \prod_{i=1,\dots k(f)} A(Di, z) =$

$$t^{r(D)-k(f)} \sum_{g\in T(D,f)} \langle D \mid f \rangle \prod_{i=1,\dots k(f)} \langle Di \mid gi \rangle\, t^{w(Di_{gi,1})} (-t)^{-w(Di_{gi,2})} (-1)^{r(Di_{gi,2})}$$

Clearly for $h = u(f, gi(i=1, \dots k(f))) = u(f, g)$:

$$\sum_{i=1,\dots k(f)} w(Di_{gi,1}) = w_o(D, h), \sum_{i=1,\dots k(f)} w(Di_{gi,2}) = w_e(D, h)$$

and

$$\sum_{i=1,\dots k(f)} r(Di_{gi,2}) = r_e(D, h)\,.$$

Moreover, since h is easily seen to be compatible with f and the gi, it follows from the Unification Lemma that $<D \mid f> \prod_{i=1,\dots k(f)} <Di \mid gi> = <D \mid h>$. Hence

$$<D \mid f> \prod_{i=1,\dots k(f)} A(Di, z) =$$
$$t^{r(D)-k(f)} \sum_{g \in T(D,f),\, h=u(f,g)} <D \mid h>\, t^{w_o(D,h)} (-t)^{-w_e(D,h)} (-1)^{r_e(D,h)}$$

It now follows from Proposition 9 that

$$H(D, z, a) = \sum_{f \in CL(D)} \sum_{g \in T(D,f),\, h=u(f,g)}$$
$$<D \mid h>\, (t^{-1}a)^{-r(D)+k(f)} \left((a-a^{-1})z^{-1}\right)^{k(f)-1} t^{w_o(D,h)} (-t)^{-w_e(D,h)} (-1)^{r_e(D,h)}$$

In this expression $k(f)$ can be replaced by $k'(h)$. Moreover u clearly defines a bijection from the set $\{(f, g)/f \in CL(D),\ g \in T(D, f)\}$ to $C'L(D)$. This completes the proof.

4. Concluding remarks

1. It would be interesting to generalize the results of Section 3 to arbitrary diagrams. This is done for Proposition 6 in a joint paper with Louis Kauffman (in preparation).

(2) Since the topological and algebraic aspects of the Alexander polynomial are well understood, one may try to use Proposition 9 to gain some insight of the same kind on the homfly polynomial. Clearly one can combine Proposition 9 with classical results which relate the polynomial $A(D, z)$ to Burau matrices, Seifert surfaces and matrices, presentations of the fundamental group of the complement... This leads to corresponding complex labelled structures which seem to be worth studying. As for the combinatorial aspects Proposition 12 is only a first step and some further progress closely connected with Proposition 9 is reported in the following forthcoming papers: [A] Circuit partitions and the homfly polynomial of closed braids, Trans. AMS, to appear, [B] A combinatorial model for the homfly polynomial, preprint.

(3) We believe that our approach to the Morton-Franks-Williams inequalities could be developed to yield more precise results on the cases of equality (which appear to be extremely frequent). For instance it is shown in [A], Proposition 7 that the first (respectively: third) inequality of Proposition 10 holds with equality if all vertices of D have negative (respectively: positive) sign.

(4) Proposition 12 can be interpreted in terms of explicit matrix representations of Hecke algebras. The homfly polynomial of a braid diagram appears as the trace of a matrix indexed by its labellings. This matrix can be computed as a product consisting of one matrix for each crossing (which incorporates the interaction and writhe contributions) and a final diagonal matrix which assigns suitable weights to the various labellings.

(5) The known relationship of the Jones sequence of state models with solutions of the Yang-Baxter equation suggests that theoretical physics might provide an interpretation of the properties of the product operation for homfly polynomials. An algebraic generalization of this operation is given by V. G. Turaev in: Algebras of loops on surfaces, algebras of knots, and quantization, LOMI preprint E-10-88.

REFERENCES

[1] ALEXANDER, J. W. Topological invariants of knots and links. *Trans. Am. Math. Soc. 30* (1928), 275-306.

[2] BAXTER, R. J. *Exactly solved models in statistical mechanics.* Academic Press, 1982.

[3] CONWAY, J. H. An enumeration of knots and links and some of their algebraic properties. *Computational problems in abstract algebra.* Pergamon Press, New York (1970), 329-358.

[4] FRANKS, J. and R. F. WILLIAMS. Braids and the Jones polynomial. *Trans. Am. Math. Soc. 303* (1987), 97-108.

[5] FREYD, P., D. YETTER, J. HOSTE, W. B. R. LICKORISH, K. MILLETT and A. OCNEANU. A new polynomial invariant of knots and links. *Bull. Am. Math. Soc. 12* (1985), 239-246.

[6] FREYD, P. and D. YETTER. A new invariant for knots and links. Preprint, University of Pennsylvania, 1984.

[7] DE LA HARPE, P., M. KERVAIRE and C. WEBER. On the Jones polynomial. *L'Enseignement Mathématique 32* (1986), 271-335.

[8] HOSTE, J. A polynomial invariant of knots and links. *Pacific Journal of Math. 124* (1986), 295-320.

[9] JAEGER, F. On Tutte polynomials and cycles of plane graphs. *J. Combinatorial Theory (B) 44* (1988), 127-146.

[10] —— Tutte polynomials and link polynomials. *Proc. Am. Math. Soc. 103* (1988), 647-654.

[11] JONES, V. F. R. A polynomial invariant for knots via Von Neumann algebras. *Bull. Am. Math. Soc. 12* (1985), 103-111.
[12] —— A new knot polynomial and Von Neumann algebras. *Notices of the Am. Math. Soc. 33* (1986), 219-225.
[13] —— Hecke algebra representations of braid groups and link polynomials. *Annals of Math. 126* (1987), 335-388.
[14] —— Notes on a talk in Atiyah's seminar, November 1986.
[15] KAUFFMAN, L. H. The Conway polynomial. *Topology 20* (1981), 101-108.
[16] —— *Formal knot theory.* Princeton university press, Mathematical notes 30, 1983.
[17] —— State models and the Jones polynomial. *Topology 26* (1987), 395-407.
[18] —— New invariants in the theory of knots. *Amer. Math. Monthly 95 n° 3* (1988), 195-242.
[19] —— An invariant of regular isotopy. *Trans. AMS.* To appear.
[20] —— State models for knot polynomials — an introduction. *Proceedings of the 1987 Winter Meeting of the Brasilian Mathematical Society.*
[21] LICKORISH, W. B. R. and K. MILLETT. A polynomial invariant of oriented links. *Topology 26* (1987), 107-141.
[22] MORTON, H. R. Seifert circles and knot polynomials. *Math. Proc. Camb. Phil. Soc. 99* (1986), 107-109.
[23] MURAKAMI, H. A formula for the two-variable link polynomial. *Topology 26* (1987), 409-412.
[24] MURASUGI, K. Jones polynomials and classical conjectures in knot theory. *Topology 26* (1987), 187-194.
[25] OCNEANU, A. A polynomial invariant for knots: a combinatorial and an algebraic approach. Preprint, University of California, Berkeley, 1985.
[26] PRZYTYCKI, J. H. and P. TRACZYK. Invariants of links of Conway type. *Kobe J. Math. 4* (1987), 115-139.
[27] THISTLETHWAITE, M. A spanning tree expansion of the Jones polynomial. *Topology 26* (1987), 297-309.
[28] TURAEV, V. G. The Yang-Baxter equation and invariants of links. *Invent. Math. 92* (1988), 527-553.
[29] TUTTE, W. T. A ring in Graph Theory. *Proc. Cambridge Phil. Soc. 43* (1947), 26-40.
[30] —— A contribution to the theory of chromatic polynomials. *Canad. J. Math. 6* (1954), 80-91.
[31] VOGEL, P. Représentations et traces des algèbres de Hecke, polynôme de Jones-Conway. *L'Enseignement Mathématique 34* (1988), 333-356.

(Reçu le 12 septembre 1988)

François Jaeger

Laboratoire de Structures Discrètes
IMAG, BP 53 X, 38041 Grenoble Cédex
France

vide-leer-empty