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and similarly with X replaced by U and X_α replaced by $U \cap X_\alpha$. Using this and the exact sequence 4.2 we get that

$$\lim_{\rightarrow} D^c_\bullet(X_\alpha, U \cap X_\alpha; \mathbf{C}) = D^c_\bullet(X, U; \mathbf{C})$$

from which the result follows by passing to homology. Q.E.D.

Let us also notice that in case X is the disjoint union of a family (X_α) of open subsets we have that

(4.9) $$\bigoplus_\alpha H^c_\bullet(X_\alpha, U \cap X_\alpha; \mathbf{C}) \xrightarrow{\sim} H^c_\bullet(X, U; \mathbf{C}).$$

5. Stokes formula

Let us consider the open subset U of the n-dimensional smooth manifold X and the resulting exact sequences

(5.1)
$$\begin{array}{c} \to H^c_p(X, \mathbf{C}) \to H^c_p(X, U; \mathbf{C}) \xrightarrow{b} H^c_{p-1}(U, \mathbf{C}) \xrightarrow{j_*} H^c_{p-1}(X, \mathbf{C}) \to \\ \leftarrow H^p(X, \mathbf{C}) \leftarrow H^p(X, U; \mathbf{C}) \xleftarrow{\partial} H^{p-1}(U, \mathbf{C}) \xleftarrow{j^*} H^{p-1}(X, \mathbf{C}) \leftarrow \end{array}$$

where the first is discussed in the previous section and the second is the sheaf cohomology sequence. The relative term in the second sequence is often written

(5.2) $$H^p_Z(X, \mathbf{C}) = H^p(X, U; \mathbf{C}), \quad Z = X - U.$$

We can now extend the biduality theorem (2.1).

(5.3) Theorem. *The cohomology sequence above is dual to the homology sequence. In particular we have a Stoke's formula*

$$\langle b\alpha, \omega \rangle = \langle \alpha, \partial\omega \rangle$$

for $\alpha \in H^c_p(X, U; \mathbf{C})$ *and* $\omega \in H^{p-1}(U, \mathbf{C})$.

Proof. The first sequence arises from the following short exact sequence of complexes, compare (4.2) and (4.7),

$$0 \to \Gamma_c(U, \Omega^{\bullet\vee}) \xrightarrow{j_*} \Gamma_c(X, \Omega^{\bullet\vee}) \to \Gamma_c(Z, \Omega^{\bullet\vee}) \to 0.$$

In order to calculate the second sequence we depart from the flabby resolution $\Omega^{\bullet\vee\vee}$ of $\mathbf{R}$ established in the proof of the biduality theorem (2.1).

The basic philosofy being that flabby sheaves are acyclic for local cohomology, [5] II. 9.3. Thus we can calculate the cohomology sequence (5.1) from the short exact sequence

$$0 \leftarrow \Gamma(U, \Omega^{\cdot \vee \vee}) \overset{j^*}{\leftarrow} \Gamma(X, \Omega^{\cdot \vee \vee}) \leftarrow \Gamma_Z(X, \Omega^{\cdot \vee \vee}) \leftarrow 0 .$$

According to formula (2.4) we may identify the arrow marked j^* with the linear dual of the arrow marked j_*. Simple evaluation according to (2.4) will be written

$$< T, l > , \quad T \in \Gamma_c(X,)^{\cdot \vee}) , \quad l \in \Gamma(X, \Omega^{\cdot \vee \vee}) .$$

This notation is compatible with the symbol introduced in section 1 taking the biduality morphism (2.6) into account. We leave the remaining details with the reader. Q.E.D.

6. Poincaré duality

Let X be a n-dimensional *oriented* smooth manifold. A compactly supported $(n-p)$-form α on X defines a compact p-chain $P\alpha$ given by

$$< P\alpha, \beta > = \int_X \alpha \wedge \beta , \quad \beta \in \Gamma(X, \Omega^p) . \tag{6.1}$$

(6.2) Theorem. *For a smooth oriented n-dimensional manifold X, the transformation P induces an isomorphism*

$$P : H_c^{n-p}(X, \mathbf{C}) \to H_p^c(X, \mathbf{C}) , \quad p \in \mathbf{N} ,$$

from de Rham cohomology with compact support to de Rham homology.

Proof. The following diagram is commutative

$$\begin{array}{ccc} \Gamma_c(X, \Omega^{n-p}) & \overset{P}{\to} & D_p^c(X, \mathbf{C}) \\ \downarrow (-1)^n d & & \downarrow (-1)^{p-1} b \\ \Gamma_c(X, \Omega^{n-p+1}) & \overset{P}{\to} & D_{p-1}^c(X, \mathbf{C}) \end{array} \tag{6.3}$$

as it follows from the relation

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{n-p} \alpha \wedge d\beta , \quad \alpha \in \Gamma_c(X, \Omega^{n-p}), \beta \in \Gamma(X, \Omega^p) ,$$