

Zeitschrift: L'Enseignement Mathématique
Band: 35 (1989)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CAUCHY RESIDUES AND DE RHAM HOMOLOGY
Kapitel: 5. Stokes formula
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DOI: <https://doi.org/10.5169/seals-57358>

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and similarly with X replaced by U and X_α replaced by $U \cap X_\alpha$. Using this and the exact sequence 4.2 we get that

$$\lim_{\rightarrow} D_c^c(X_\alpha, U \cap X_\alpha; \mathbf{C}) = D_c^c(X, U; \mathbf{C})$$

from which the result follows by passing to homology. Q.E.D.

Let us also notice that in case X is the disjoint union of a family (X_α) of open subsets we have that

$$(4.9) \quad \bigoplus_{\alpha} H_c^c(X_\alpha, U \cap X_\alpha; \mathbf{C}) \xrightarrow{\sim} H_c^c(X, U; \mathbf{C}).$$

5. STOKES FORMULA

Let us consider the open subset U of the n -dimensional smooth manifold X and the resulting exact sequences

$$(5.1) \quad \begin{array}{ccccccc} \rightarrow & H_p^c(X, \mathbf{C}) & \rightarrow & H_p^c(X, U; \mathbf{C}) & \xrightarrow{b} & H_{p-1}^c(U, \mathbf{C}) & \xrightarrow{j_*} & H_{p-1}^c(X, \mathbf{C}) & \rightarrow \\ \leftarrow & H^p(X, \mathbf{C}) & \leftarrow & H^p(X, U; \mathbf{C}) & \xleftarrow{\partial} & H^{p-1}(U, \mathbf{C}) & \xleftarrow{j^*} & H^{p-1}(X, \mathbf{C}) & \leftarrow \end{array}$$

where the first is discussed in the previous section and the second is the sheaf cohomology sequence. The relative term in the second sequence is often written

$$(5.2) \quad H_Z^p(X, \mathbf{C}) = H^p(X, U; \mathbf{C}), \quad Z = X - U.$$

We can now extend the biduality theorem (2.1).

(5.3) THEOREM. *The cohomology sequence above is dual to the homology sequence. In particular we have a Stoke's formula*

$$\langle b\alpha, \omega \rangle = \langle \alpha, \partial\omega \rangle$$

for $\alpha \in H_p^c(X, U; \mathbf{C})$ and $\omega \in H^{p-1}(U, \mathbf{C})$.

Proof. The first sequence arises from the following short exact sequence of complexes, compare (4.2) and (4.7),

$$0 \rightarrow \Gamma_c(U, \Omega^{\bullet \vee}) \xrightarrow{j_*} \Gamma_c(X, \Omega^{\bullet \vee}) \rightarrow \Gamma_c(Z, \Omega^{\bullet \vee}) \rightarrow 0.$$

In order to calculate the second sequence we depart from the flabby resolution $\Omega^{\bullet \vee \vee}$ of \mathbf{R} established in the proof of the biduality theorem (2.1).

The basic philosophy being that flabby sheaves are acyclic for local cohomology, [5] II. 9.3. Thus we can calculate the cohomology sequence (5.1) from the short exact sequence

$$0 \leftarrow \Gamma(U, \Omega^{\bullet \vee \vee}) \xleftarrow{j^*} \Gamma(X, \Omega^{\bullet \vee \vee}) \leftarrow \Gamma_Z(X, \Omega^{\bullet \vee \vee}) \leftarrow 0.$$

According to formula (2.4) we may identify the arrow marked j^* with the linear dual of the arrow marked j_* . Simple evaluation according to (2.4) will be written

$$\langle T, l \rangle, \quad T \in \Gamma_c(X, \Omega^{\bullet \vee}), \quad l \in \Gamma(X, \Omega^{\bullet \vee \vee}).$$

This notation is compatible with the symbol introduced in section 1 taking the biduality morphism (2.6) into account. We leave the remaining details with the reader. Q.E.D.

6. POINCARÉ DUALITY

Let X be a n -dimensional *oriented* smooth manifold. A compactly supported $(n-p)$ -form α on X defines a compact p -chain $P\alpha$ given by

$$(6.1) \quad \langle P\alpha, \beta \rangle = \int_X \alpha \wedge \beta, \quad \beta \in \Gamma(X, \Omega^p).$$

(6.2) **THEOREM.** *For a smooth oriented n -dimensional manifold X , the transformation P induces an isomorphism*

$$P: H_c^{n-p}(X, \mathbf{C}) \rightarrow H_p^c(X, \mathbf{C}), \quad p \in \mathbf{N},$$

from de Rham cohomology with compact support to de Rham homology.

Proof. The following diagram is commutative

$$(6.3) \quad \begin{array}{ccc} \Gamma_c(X, \Omega^{n-p}) & \xrightarrow{P} & D_p^c(X, \mathbf{C}) \\ \downarrow (-1)^n d & & \downarrow (-1)^{p-1} b \\ \Gamma_c(X, \Omega^{n-p+1}) & \xrightarrow{P} & D_{p-1}^c(X, \mathbf{C}) \end{array}$$

as it follows from the relation

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{n-p} \alpha \wedge d\beta, \quad \alpha \in \Gamma_c(X, \Omega^{n-p}), \beta \in \Gamma(X, \Omega^p),$$