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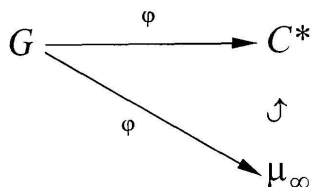
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$$c(\rho \circ f) = f^*(c.(\rho)).$$

CH2. $c.(\rho_1 \oplus \rho_2) = c.(\rho_1) \cdot c.(\rho_2).$

CH3. $c_1: \text{Hom}(G, \mathbf{C}^*) \rightarrow H^2(G, \mathbf{Z})$ is an isomorphism and can be described as follows: For $\varphi \in \text{Hom}(G, \mathbf{C}^*)$, let φ also denote its unique factorization



Now $c_1(\varphi) = \varphi^*(u).$

Remark. As shown in [7], CH1, CH2 and CH3 uniquely determine the Chern classes defined by u . As different choices of u clearly defines different Chern classes (just observe that

$$H^2(\mu_\infty, \mathbf{Z}) \cong \varinjlim H^2(G_i, \mathbf{Z}),$$

the limit taken over all finite cyclic subgroups), there is a one-to-one correspondence between Chern classes on finite groups and $\hat{\mathbf{Z}}$ generators of $H^2(\mu_\infty, \mathbf{Z})$.

This paper has been organized as follows.

Theorem 2 is proved in Section 1, Theorem 4 in Section 2, and Theorem 5 in Section 3. Proposition 3 i) was proved in [7], and the remaining part of this proposition can be obtained similarly.

Finally, in Section 4 it is shown that there exists a very simple extension of the theory of Chern classes on finite groups to locally finite groups.

I would like to thank Jørgen Tornehave for a helpful conversation.

SECTION 1. PROOF OF THEOREM 2

CH1 is quite trivial, so let me first prove CH2. Let $\dim \rho_i = n_i$, $\dim \rho = n$, so that $n_1 + n_2 = n$. By assumption, ρ factors through the parabolic subgroup $P = P(k_p)$

$$P = \begin{pmatrix} & n_1 & & n_2 \\ * & & * & \\ & & & \\ 0 & & & * \end{pmatrix}$$

which is isomorphic to a semi-direct product of $Gl_{n_1}(\bar{k}_p) \times Gl_{n_2}(\bar{k}_p)$ acting on a unipotent subgroup U .

As U is a direct limit of p -groups,

$$H^k(U, \hat{\mathbf{Z}}_l) = 0 \quad \text{for } k > 0.$$

Thus

$$\begin{aligned} H^*(P, \hat{\mathbf{Z}}_l) &\cong H^*(Gl_{n_1}(\bar{k}_p), \hat{\mathbf{Z}}_l) \otimes H^*(Gl_{n_2}(\bar{k}_p), \hat{\mathbf{Z}}_l) \\ &\cong P(\alpha_1, \dots, \alpha_{n_1}) \otimes P(\beta_1, \dots, \beta_{n_2}). \end{aligned}$$

Let

$$H^*(Gl_n(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(\sigma_1, \dots, \sigma_n)$$

and

$$H^*(T_n(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(x_1, \dots, x_n).$$

As $T_n(\bar{k}_p) \cong T_{n_1}(\bar{k}_p) \times T_{n_2}(\bar{k}_p)$, I shall consider

$$H^*(T_{n_1}(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(x_1, \dots, x_{n_1})$$

and

$$H^*(T_{n_2}(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(x_{n_1+1}, \dots, x_n)$$

as contained in $H^*(T_n(\bar{k}_p), \hat{\mathbf{Z}}_l)$. Furthermore, as all restriction maps are injective, I shall view $H^*(Gl_n(\bar{k}_p), \hat{\mathbf{Z}}_l)$ and $H^*(Gl_{n_i}(\bar{k}_p), \hat{\mathbf{Z}}_l)$, $i = 1, 2$, as subspaces of $H^*(T_n(\bar{k}_p), \hat{\mathbf{Z}}_l)$. Thus

α_i = the i 'th elementary symmetric polynomial in x_1, \dots, x_{n_1}

β_i = the i 'th elementary symmetric polynomial in x_{n_1+1}, \dots, x_n

σ_i = the i 'th elementary symmetric polynomial in x_1, \dots, x_n .

Furthermore, the formula

$$c \cdot (\rho_1 \oplus \rho_2) = c \cdot (\rho_1 \oplus \rho_2)$$

is equivalent to

$$1 + \sigma_1 t + \dots + \sigma_n t^n = (1 + \alpha_1 t + \dots + \alpha_{n_1} t^{n_1}) \cdot (1 + \beta_1 t + \dots + \beta_{n_2} t^{n_2}),$$

and this follows from the identity

$$\begin{aligned} \sum_{i=0}^n \sigma_i t^i &= \prod_{i=1}^n (1 + tx_i) = \prod_{i=1}^{n_1} (1 + tx_i) \cdot \prod_{i=n_1+1}^{n_2} (1 + tx_i) \\ &= \left(\sum_{i=0}^{n_1} \alpha_i t^i \right) \left(\sum_{i=0}^{n_2} \beta_i \cdot t^i \right). \end{aligned}$$

To prove CH3, observe that for G locally finite the homology groups $H_i(G, \mathbf{Z})$ are all torsion groups for $i > 0$ as

$$H_i(G, \mathbf{Z}) \cong \varinjlim H_i(G_k, \mathbf{Z}),$$

the limit taken over a family of finite subgroups G_k of G such that $\varinjlim G_k = G$. Now, by the universal coefficient theorem,

$$0 \rightarrow \text{Ext}_{\mathbf{Z}}^1(H_1(G, \mathbf{Z}), \hat{\mathbf{Q}}_l) \rightarrow H^2(G, \hat{\mathbf{Q}}_l) \rightarrow \text{Hom}_{\mathbf{Z}}(H_2(G, \mathbf{Z}), \hat{\mathbf{Q}}_l) \rightarrow 0$$

is exact ($\hat{\mathbf{Q}}_l$ is the quotient field of $\hat{\mathbf{Z}}_l$) so it follows that $H^2(G, \hat{\mathbf{Q}}_l) = 0$ as $\hat{\mathbf{Q}}_l$ is both torsion-free and divisible. From the long exact sequence in cohomology it now follows that

$$H^1(G, \hat{\mathbf{Q}}_l/\hat{\mathbf{Z}}_l) \cong H^2(G, \hat{\mathbf{Z}}_l).$$

Finally, as $\hat{\mathbf{Q}}_l/\mathbf{Z}_l \cong C_{l^\infty}$, where C_{l^∞} is the injective hull of a cyclic l -group, it follows that

$$\prod_{l \neq p} H^2(G, \hat{\mathbf{Z}}_l) \cong \prod_{l \neq p} H^1(G, C_{l^\infty}) \cong H^1(G, \prod_{l \neq p} C_{l^\infty}) = H^1(G, \bigoplus_{l \neq p} C_{l^\infty}).$$

The last equality holds, as G is locally finite and $\bigoplus_{l \neq p} C_{l^\infty}$ is the torsion subgroup of $\prod_{l \neq p} C_{l^\infty}$.

SECTION 2. PROOF OF THEOREM 4

Let G be a given finite group of order $|G|$ and

$$\rho: G \rightarrow Gl_n(\mathbf{C})$$

a complex representation.

Choose q to be a power of a prime number p different from l such that

$$q \equiv 1 \pmod{|G|}$$

Define

$$\phi: Gl_n(q) \rightarrow \mathbf{C}$$

by

$$\phi(g) = \sum_{i=1}^n e_p(\lambda_i)$$