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**Kapitel:** Section 1. Proof of Theorem 2

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$$c(\rho\circ f) = f^*(c.(\rho)).$$

*CH2.* $c.(\rho_1 \oplus \rho_2) = c.(\rho_1)\cdot c.(\rho_2)$.

*CH3.* $c_1 : \mathrm{Hom}(G, \mathbf{C}^*) \to H^2(G, \mathbf{Z})$ *is an isomorphism and can be described as follows: For* $\varphi \in \mathrm{Hom}(G, \mathbf{C}^*)$, *let* $\varphi$ *also denote its unique factorization*

$$\begin{array}{ccc} G & \xrightarrow{\ \varphi\ } & C^* \\ & \searrow^{\varphi} & \circlearrowright \\ & & \mu_\infty \end{array}$$

*Now* $c_1(\varphi) = \varphi^*(u)$.

*Remark.* As shown in [7], CH1, CH2 and CH3 uniquely determine the Chern classes defined by $u$. As different choices of $u$ clearly defines different Chern classes (just observe that

$$H^2(\mu_\infty, \mathbf{Z}) \cong \varinjlim H^2(G_i, \mathbf{Z}),$$

the limit taken over all finite cyclic subgroups), there is a one-to-one correspondence between Chern classes on finite groups and $\hat{\mathbf{Z}}$ generators of $H^2(\mu_\infty, \mathbf{Z})$.

This paper has been organized as follows.

Theorem 2 is proved in Section 1, Theorem 4 in Section 2, and Theorem 5 in Section 3. Proposition 3 i) was proved in [7], and the remaining part of this proposition can be obtained similarly.

Finally, in Section 4 it is shown that there exists a very simple extension of the theory of Chern classes on finite groups to locally finite groups.

I would like to thank Jørgen Tornehave for a helpful conversation.

## Section 1. Proof of Theorem 2

CH1 is quite trivial, so let me first prove CH2. Let $\dim \rho_i = n_i$, $\dim \rho = n$, so that $n_1 + n_2 = n$. By assumption, $\rho$ factors through the parabolic subgroup $P = P(\bar{k}_p)$

$$P = \left\{ \begin{array}{cc} \overset{n_1}{*} & \overset{n_2}{*} \\ 0 & * \end{array} \right\}$$

which is isomorphic to a semi-direct product of $Gl_{n_1}(\bar{k}_p) \times Gl_{n_2}(\bar{k}_p)$ acting on a unipotent subgroup $U$.

As $U$ is a direct limit of $p$-groups,

$$H^k(U, \hat{\mathbf{Z}}_l) = 0 \quad \text{for } k > 0 .$$

Thus

$$\begin{aligned} H^*(P, \hat{\mathbf{Z}}_l) &\cong H^*(Gl_{n_1}(\bar{k}_p), \hat{\mathbf{Z}}_l) \otimes H^*(Gl_{n_2}(\bar{k}_p), \hat{\mathbf{Z}}_l) \\ &\cong P(\alpha_1, \ldots, \alpha_{n_1}) \otimes P(\beta_1, \ldots, \beta_{n_2}) . \end{aligned}$$

Let

$$H^*(Gl_n(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(\sigma_1, \ldots, \sigma_n)$$

and

$$H^*(T_n(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(x_1, \ldots, x_n) .$$

As $T_n(\bar{k}_p) \cong T_{n_1}(\bar{k}_p) \times T_{n_2}(\bar{k}_p)$, I shall consider

$$H^*(T_{n_1}(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(x_1, \ldots, x_{n_1})$$

and

$$H^*(T_{n_2}(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(x_{n_1+1}, \ldots, x_n)$$

as contained in $H^*(T_n(\bar{k}_p), \hat{\mathbf{Z}}_l)$. Furthermore, as all restriction maps are injective, I shall view $H^*(Gl_n(\bar{k}_p), \hat{\mathbf{Z}}_l)$ and $H^*(Gl_{n_i}(\bar{k}_p), \hat{\mathbf{Z}}_l)$, $i = 1, 2$, as subspaces of $H^*(T_n(\bar{k}_p), \hat{\mathbf{Z}}_l)$. Thus

$\alpha_i$ = the $i$'th elementary symmetric polynomial in $x_1, \ldots, x_{n_1}$

$\beta_i$ = the $i$'th elementary symmetric polynomial in $x_{n_1+1}, \ldots, x_n$

$\sigma_i$ = the $i$'th elementary symmetric polynomial in $x_1, \ldots, x_n$.

Furthermore, the formula

$$c.(\rho_1 \oplus \rho_2) = c.(\rho_1 \oplus \rho_2)$$

is equivalent to

$$1 + \sigma_1 t + \ldots + \sigma_n t^n = (1 + \alpha_1 t + \ldots + \alpha_{n_1} t^{n_1}) \cdot (1 + \beta_1 t + \ldots + \beta_{n_2} t^{n_2}) ,$$

and this follows from the identity

$$\begin{aligned} \sum_{i=0}^{n} \sigma_i t^i &= \prod_{i=1}^{n} (1 + t x_i) = \prod_{i=1}^{n_1} (1 + t x_i) \cdot \prod_{i=n_1+1}^{n_2} (1 + t x_i) \\ &= \left( \sum_{i=0}^{n_1} \alpha_i t^i \right) \left( \sum_{i=0}^{n_2} \beta_i \cdot t^i \right) . \end{aligned}$$

To prove CH3, observe that for $G$ locally finite the homology groups $H_i(G, \mathbf{Z})$ are all torsion groups for $i > 0$ as

$$H_i(G, \mathbf{Z}) \cong \varinjlim H_i(G_k, \mathbf{Z}),$$

the limit taken over a family of finite subgroups $G_k$ of $G$ such that $\varinjlim G_k = G$. Now, by the universal coefficient theorem,

$$0 \to \mathrm{Ext}^1_{\mathbf{Z}}(H_1(G, \mathbf{Z}), \hat{\mathbf{Q}}_l) \to H^2(G, \hat{\mathbf{Q}}_l) \to \mathrm{Hom}_{\mathbf{Z}}(H_2(G, \mathbf{Z}), \hat{\mathbf{Q}}_l) \to 0$$

is exact ($\hat{\mathbf{Q}}_l$ is the quotient field of $\hat{\mathbf{Z}}_l$) so it follows that $H^2(G, \hat{\mathbf{Q}}_l) = 0$ as $\hat{\mathbf{Q}}_l$ is both torsion-free and divisible. From the long exact sequence in cohomology it now follows that

$$H^1(G, \hat{\mathbf{Q}}_l/\hat{\mathbf{Z}}_l) \cong H^2(G, \hat{\mathbf{Z}}_l).$$

Finally, as $\hat{\mathbf{Q}}_l/\mathbf{Z}_l \cong C_{l^\infty}$, where $C_{l^\infty}$ is the injective hull of a cyclic $l$-group, it follows that

$$\prod_{l \neq p} H^2(G, \hat{\mathbf{Z}}_l) \cong \prod_{l \neq p} H^1(G, C_{l^\infty}) \cong H^1(G, \prod_{l \neq p} C_{l^\infty}) = H^1(G, \bigoplus_{l \neq p} C_{l^\infty}).$$

The last equality holds, as $G$ is locally finite and $\bigoplus_{l \neq p} C_{l^\infty}$ is the torsion subgroup of $\prod_{l \neq p} C_{l^\infty}$.

## Section 2. Proof of Theorem 4

Let $G$ be a given finite group of order $|G|$ and

$$\rho: G \to Gl_n(\mathbf{C})$$

a complex representation.

Choose $q$ to be a power of a prime number $p$ different from $l$ such that

$$q \equiv 1 \text{ (modulo } |G|)$$

Define

$$\phi: Gl_n(q) \to \mathbf{C}$$

by

$$\phi(g) = \sum_{i=1}^{n} e_p(\lambda_i)$$