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Autor: Kroll, Ove

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To prove CH3, observe that for G locally finite the homology groups $H_i(G, \mathbf{Z})$ are all torsion groups for $i > 0$ as

$$H_i(G, \mathbf{Z}) \cong \varinjlim H_i(G_k, \mathbf{Z}),$$

the limit taken over a family of finite subgroups G_k of G such that $\varinjlim G_k = G$. Now, by the universal coefficient theorem,

$$0 \to \operatorname{Ext}^1_{\mathbf{Z}}(H_1(G, \mathbf{Z}), \hat{\mathbf{Q}}_l) \to H^2(G, \hat{\mathbf{Q}}_l) \to \operatorname{Hom}_{\mathbf{Z}}(H_2(G, \mathbf{Z}), \hat{\mathbf{Q}}_l) \to 0$$

is exact ($\hat{\mathbf{Q}}_l$ is the quotient field of $\hat{\mathbf{Z}}_l$) so it follows that $H^2(G, \hat{\mathbf{Q}}_l) = 0$ as $\hat{\mathbf{Q}}_l$ is both torsion-free and divisible. From the long exact sequence in cohomology it now follows that

$$H^1(G, \hat{\mathbf{Q}}_l/\hat{\mathbf{Z}}_l) \cong H^2(G, \hat{\mathbf{Z}}_l).$$

Finally, as $\hat{\mathbf{Q}}_l/\mathbf{Z}_l \cong C_{l^\infty}$, where C_{l^∞} is the injective hull of a cyclic l-group, it follows that

$$\prod_{l \neq p} H^2(G, \hat{\mathbf{Z}}_l) \cong \prod_{l \neq p} H^1(G, C_{l^\infty}) \cong H^1(G, \prod_{l \neq p} C_{l^\infty}) = H^1(G, \bigoplus_{l \neq p} C_{l^\infty}).$$

The last equality holds, as G is locally finite and $\bigoplus_{l \neq p} C_{l^\infty}$ is the torsion subgroup of $\prod_{l \neq p} C_{l^\infty}$.

Section 2. Proof of Theorem 4

Let G be a given finite group of order $|G|$ and

$$\rho: G \to Gl_n(\mathbf{C})$$

a complex representation.

Choose q to be a power of a prime number p different from l such that

$$q \equiv 1 \ (\text{modulo } |G|)$$

Define

$$\phi: Gl_n(q) \to \mathbf{C}$$

by

$$\phi(g) = \sum_{i=1}^{n} e_p(\lambda_i)$$

where $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of g. As shown by J. A. Green in [4], ϕ is a virtual complex character of $Gl_n(q)$.

Furthermore let

$$f : G \to Gl_n(q)$$

be the mod-p reduction of ρ to $Gl_n(q)$. (It factors through $Gl_n(q)$, as all $|G|$-roots of unity are contained in the Galois field $GF(q)$ with q elements).

Let $f^*: R_{\mathbf{C}}(Gl_n(q)) \to R_{\mathbf{C}}(G)$ be a map induced on complex character rings by f. By inspection

$$f^*(\phi) = \rho .$$

Let $a = \nu_l(q-1)$, where ν_l is the l-adic valuation and let

$$p^{**} : H^{**}(G, \hat{\mathbf{Z}}_l) \to H^{**}(G, \mathbf{Z}_{l^a})$$

be the map induced by the projection $p: \hat{\mathbf{Z}}_l \to \mathbf{Z}_{l^a}$. Clearly p^{**} is injective in positive dimensions, as multiplication by l^a is zero on $H^{**}(G, \hat{\mathbf{Z}}_l)$.

Now the following diagram is commutative

$$\begin{array}{ccc}
H^{**}(G, \hat{\mathbf{Z}}_l) & \overset{p^{**}}{\rightarrow} & H^{**}(G, \mathbf{Z}_{l^a}) \\
f^{**} \uparrow & & \uparrow f^{**} \\
H^{**}(Gl_n(q), \hat{\mathbf{Z}}_l) & \rightarrow & H^{**}(Gl_n(q), \mathbf{Z}_{l^a}) \\
\downarrow \text{res} & & \downarrow \text{res} \\
H^{**}(T_n(q), \hat{\mathbf{Z}}_l) & \overset{p^{**}}{\rightarrow} & H^{**}(T_n(q), \mathbf{Z}_{l^a})
\end{array}$$

where the restriction map on the right is injective as shown in [6].

Thus for $i = 1, 2$

$$c_{\bullet}^{(i)}(d_{p_i}(\rho)) = (p^{**})^{-1} f^{**}(\text{res})^{-1} p^{**}(d_{p_i}(\tau))$$

where τ is the restriction of the virtual character ϕ to $T_n(q)$. [Note that $(p^{**})^{-1}$ and $(\text{res})^{-1}$ both make sense as the above diagram is commutative].

Thus to show equality, it suffices using CH1 in Theorem 2, to show that

$$c_{\bullet}^{(1)}(d_{p_1}(\tau)) = c_{\bullet}^{(2)}(d_{p_2}(\tau))$$

But $T_n(q)$ is abelian so τ is a direct sum of n one-dimensional representations. By CH2 of Theorem 2 it suffices to show that for a one dimensional representation

$$\varphi\colon T_n(q) \to \mu_\infty ,$$
$$c_\bullet^{(1)}\big(d_{p_1}(\varphi)\big) = c_\bullet^{(2)}\big(d_{p_2}(\varphi)\big) .$$

But

$$d_{p_i}(\varphi) = e_{p_i}^{-1} \circ \varphi$$

and

$$c_\bullet^{(i)}\big(d_{p_i}(\varphi)\big) = \varphi^* \circ (e_{p_i}^{-1})^* \big(e_{p_i}^*(u)\big) = \varphi^*(u) .$$

Remark. It is necessary to reduce to $\mathbf{Z}_{l^a}$ coefficients as the restriction map

$$H^*\big(Gl_n(q), \hat{\mathbf{Z}}_l\big) \to H^*\big(T_n(q), \hat{\mathbf{Z}}_l\big)$$

is not injective in general.

SECTION 3. PROOF OF THEOREM 5

CH1 and CH2 clearly follow from resp. CH1 and CH2 in Theorem 2 together with the functoriality of the decomposition map d_p i.e. the diagram

$$\begin{array}{ccc} R_{\mathbf{C}}(G) & \xrightarrow{f^*} & R_{\mathbf{C}}(H) \\ \downarrow{\scriptstyle d_p} & & \downarrow{\scriptstyle d_p} \\ R_p(G) & \xrightarrow{f^*} & R_p(H) \end{array}$$

is commutative for a group homomorphism $f: H \to G$. To obtain CH3 note that $d_p(\varphi) = e_p^{-1} \circ \varphi$ so by definition

$$c_1(\varphi) = c_1\big(d_p(\varphi)\big) = (e_p^{-1}\circ\varphi)\,(e_p^*\, u) = \varphi^* \circ (e_p^{-1})^* \circ e_p^*(u) = \varphi^*(u) .$$

Furthermore let δ be the connecting homomorphism obtained from the exact sequence

$$\mathbf{Z} \rightarrowtail \mathbf{Q} \twoheadrightarrow \mathbf{Q}/\mathbf{Z}$$

As the diagram

$$\begin{array}{ccc} H^1(\mu_\infty, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\overset{\sim}{\delta}} & H^2(\mu_\infty, \mathbf{Z}) \\ \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \varphi^*} \\ H^1(G, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\overset{\sim}{\delta}} & H^2(G, \mathbf{Z}) \end{array}$$