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is commutative and as both δ 's are isomorphisms it suffices to show that

$$\delta^{-1} \circ c_1 : \text{Hom}(G, \mathbf{C}^*) \rightarrow H^1(G, \mathbf{Q}/\mathbf{Z}) \simeq \text{Hom}(G, \mathbf{Q}/\mathbf{Z})$$

is an isomorphism.

But by inspection $\delta^{-1} \circ c_1(\varphi) = \delta^{-1}(u) \circ \varphi$, and as u is a $\hat{\mathbf{Z}} = \text{End}_{\mathbf{Z}}(\mu_\infty)$ generator for $H^2(\mu_\infty, \mathbf{Z}) \simeq \hat{\mathbf{Z}}$, $\delta^{-1}(u)$ is an isomorphism

$$\delta^{-1}(u) : \mu_\infty \xrightarrow{\sim} \mathbf{Q}/\mathbf{Z}.$$

SECTION 4. CHERN CLASSES FOR LOCALLY FINITE GROUPS

The definition will be based on the following two observations. In the following, let $G = \varinjlim G_k$ be a locally finite group where $\{G_k\}$ is a family of finite subgroups.

LEMMA. *Let*

$$\varphi : G \rightarrow \text{Gl}_n(\mathbf{C})$$

be a representation of G . Then φ is uniquely determined by its restrictions

$$\varphi_k : G_k \rightarrow \text{Gl}_n(\mathbf{C}).$$

Conversely given a family of compatible representations $\varphi_k : G_k \rightarrow \text{Gl}_n(\mathbf{C})$, there exists a unique $\varphi : G \rightarrow \text{Gl}_n(\mathbf{C})$ which restricts to φ_k for all k .

Proof. From the universal property of the direct limit, we have

$$\text{Hom}(G, \text{Gl}_n(\mathbf{C})) \cong \varprojlim \text{Hom}(G_k, \text{Gl}_n(\mathbf{C})).$$

PROPOSITION. *For all $i \geq 0$, the natural map*

$$H^i(G, \mathbf{Z}) \cong \varprojlim H^i(G_k, \mathbf{Z}).$$

is an isomorphism.

Proof. Obvious for $i = 0, 1$. For $i \geq 1$, the homology groups

$$H_i(G, \mathbf{Z}) = \varinjlim H_i(G_k, \mathbf{Z})$$

are all abelian torsion groups.

Now by the universal coefficient theorem ($i \geq 1$)

$$H^i(G, \mathbf{Z}) \cong \text{Ext}_{\mathbf{Z}}^1(H_{i-1}(G, \mathbf{Z}), \mathbf{Z})$$

and as observed above, for $i \geq 2$, $H_{i-1}(G, \mathbf{Z})$ is torsion. Thus

$$\begin{aligned} \text{Ext}_{\mathbf{Z}}^1(H_{i-1}(G, \mathbf{Z}), \mathbf{Z}) &\cong \text{Hom}_{\mathbf{Z}}(H_{i-1}(G, \mathbf{Z}), \mathbf{Q}/\mathbf{Z}) \\ &\cong \text{Hom}_{\mathbf{Z}}(\varinjlim H_{i-1}(G_k, \mathbf{Z}), \mathbf{Q}/\mathbf{Z}) \cong \varprojlim \text{Hom}_{\mathbf{Z}}(H_{i-1}(G_k, \mathbf{Z}), \mathbf{Q}/\mathbf{Z}) \\ &\simeq \varprojlim \text{Ext}_{\mathbf{Z}}^1(H_{i-1}(G_k, \mathbf{Z}), \mathbf{Z}) \cong \varprojlim H^i(G_k, \mathbf{Z}). \end{aligned}$$

Combining these two results, there exists for a representation

$$\varphi: G \rightarrow \text{Gl}_n(\mathbf{C})$$

of a locally finite group $G = \varinjlim G_k$ a unique cohomology class $c.(\varphi) \in H^{**}(G, \mathbf{Z})$ such that for all k

$$\text{res}_{G_k}^G(c.(\varphi)) = c.(\varphi_k)$$

Using this uniqueness result, it is easy to see that these classes satisfy the properties CH1, CH2 and CH3 and that they are uniquely determined by these properties.

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