

**Zeitschrift:** L'Enseignement Mathématique  
**Band:** 35 (1989)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** CAUCHY RESIDUES AND DE RHAM HOMOLOGY  
**Kapitel:** 6. POINCARÉ DUALITY  
**Autor:** Iversen, Birger  
**DOI:** <https://doi.org/10.5169/seals-57358>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

**Download PDF:** 07.10.2024

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

The basic philosophy being that flabby sheaves are acyclic for local cohomology, [5] II. 9.3. Thus we can calculate the cohomology sequence (5.1) from the short exact sequence

$$0 \leftarrow \Gamma(U, \Omega^{\bullet \vee \vee}) \xleftarrow{j^*} \Gamma(X, \Omega^{\bullet \vee \vee}) \leftarrow \Gamma_Z(X, \Omega^{\bullet \vee \vee}) \leftarrow 0.$$

According to formula (2.4) we may identify the arrow marked  $j^*$  with the linear dual of the arrow marked  $j_*$ . Simple evaluation according to (2.4) will be written

$$\langle T, l \rangle, \quad T \in \Gamma_c(X, \Omega^{\bullet \vee}), \quad l \in \Gamma(X, \Omega^{\bullet \vee \vee}).$$

This notation is compatible with the symbol introduced in section 1 taking the biduality morphism (2.6) into account. We leave the remaining details with the reader. Q.E.D.

## 6. POINCARÉ DUALITY

Let  $X$  be a  $n$ -dimensional *oriented* smooth manifold. A compactly supported  $(n-p)$ -form  $\alpha$  on  $X$  defines a compact  $p$ -chain  $P\alpha$  given by

$$(6.1) \quad \langle P\alpha, \beta \rangle = \int_X \alpha \wedge \beta, \quad \beta \in \Gamma(X, \Omega^p).$$

(6.2) **THEOREM.** *For a smooth oriented  $n$ -dimensional manifold  $X$ , the transformation  $P$  induces an isomorphism*

$$P: H_c^{n-p}(X, \mathbb{C}) \rightarrow H_p^c(X, \mathbb{C}), \quad p \in \mathbb{N},$$

*from de Rham cohomology with compact support to de Rham homology.*

*Proof.* The following diagram is commutative

$$(6.3) \quad \begin{array}{ccc} \Gamma_c(X, \Omega^{n-p}) & \xrightarrow{P} & D_p^c(X, \mathbb{C}) \\ \downarrow (-1)^n d & & \downarrow (-1)^{p-1} b \\ \Gamma_c(X, \Omega^{n-p+1}) & \xrightarrow{P} & D_{p-1}^c(X, \mathbb{C}) \end{array}$$

as it follows from the relation

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{n-p} \alpha \wedge d\beta, \quad \alpha \in \Gamma_c(X, \Omega^{n-p}), \beta \in \Gamma(X, \Omega^p),$$

using that  $\int d(\alpha \wedge \beta) = 0$ . Upon replacing  $X$  by an arbitrary open subset we obtain a morphism of complexes of sheaves

$$(6.4) \quad P: \Omega^\bullet[n] \rightarrow \Omega^{\bullet \vee}$$

with the signs of the differentials modified according to the commutative diagram (6.3). The morphism (6.4) is a quasi-isomorphism as it follows by checking the case  $X = \mathbf{R}^n$  by means of the Poincaré lemma with and without compact support. As in the proof of (2.1) we conclude that  $P$  induces a quasi-isomorphism

$$P: \Gamma_c(X, \Omega^\bullet[n]) \rightarrow \Gamma_c(X, \Omega^{\bullet \vee}).$$

The second complex may be identified with  $D^c(X, \mathbf{C})$  as we have seen in (2.3) and the result follows by passing to homology. Q.E.D.

Let us extend Poincaré duality to the relative groups of an open subset  $U$  of  $X$  with complement  $Z$  in  $X$ . With the notation of (6.1), the operator  $P$  from (6.4) induces a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \Gamma_c(U, \Omega^\bullet[n]) & \rightarrow & \Gamma_c(X, \Omega^\bullet[n]) & \rightarrow & \Gamma_c(Z, \Omega^\bullet[n]) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow P \\ 0 & \rightarrow & D^c(U, \mathbf{C}) & \xrightarrow{j_*} & D^c(X, \mathbf{C}) & \rightarrow & D^c(X, U; \mathbf{C}) \rightarrow 0. \end{array}$$

Again the differentials in the bottom row must be modified as in (6.3). The unmarked vertical arrows are the quasi-isomorphisms of Poincaré duality. The vertical arrow marked  $P$  is induced by the algebra of the diagram. Again, from algebra we deduce a *quasi-isomorphism*

$$(6.5) \quad P: \Gamma_c(Z, \Omega^\bullet[n]) \rightarrow D^c(X, U; \mathbf{C}), \quad Z = X - U.$$

Passing to homology we obtain the Poincaré duality isomorphism

$$(6.6) \quad P: H_c^{n-p}(Z, \mathbf{C}) \xrightarrow{\sim} H_p^c(X, U; \mathbf{C}).$$

The  $p$ 'th sheaf cohomology group with compact support  $H_c^p(Z, \mathbf{C})$  has the following de Rham representation

$$(6.7) \quad \left\{ \omega \in \Gamma_c(X, \Omega^p) \mid \text{Supp}(d\omega) \subseteq U \right\} \Big/ \begin{array}{l} \left\{ d\nu \mid \nu \in \Gamma_c(X, \Omega^{p-1}) \right\} \\ + \left\{ \omega \in \Gamma_c(X, \Omega^p) \mid \text{Supp}(\omega) \subseteq U \right\} \end{array}$$

as it follows from the exact sequence making up the top row of the diagram above.