Zeitschrift: L'Enseignement Mathématique
Band: 35 (1989)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CAUCHY RESIDUES AND DE RHAM HOMOLOGY
Kapitel: 6. POINCARÉ DUALITY
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DOI: https://doi.org/10.5169/seals-57358

The basic philosofy being that flabby sheaves are acyclic for local cohomology, [5] II. 9.3. Thus we can calculate the cohomology sequence (5.1) from the short exact sequence

$$0 \leftarrow \Gamma(U, \Omega^{\cdot\vee\vee}) \overset{j^*}{\leftarrow} \Gamma(X, \Omega^{\cdot\vee\vee}) \leftarrow \Gamma_Z(X, \Omega^{\cdot\vee\vee}) \leftarrow 0\,.$$

According to formula (2.4) we may identify the arrow marked j^* with the linear dual of the arrow marked j_*. Simple evaluation according to (2.4) will be written

$$<T, l>\,, \quad T \in \Gamma_c(X,)^{\cdot\vee}), \quad l \in \Gamma(X, \Omega^{\cdot\vee\vee})\,.$$

This notation is compatible with the symbol introduced in section 1 taking the biduality morphism (2.6) into account. We leave the remaining details with the reader. Q.E.D.

6. Poincaré duality

Let X be a n-dimensional *oriented* smooth manifold. A compactly supported $(n-p)$-form α on X defines a compact p-chain $P\alpha$ given by

(6.1) $$<P\alpha, \beta> \;=\; \int_X \alpha \wedge \beta\,, \quad \beta \in \Gamma(X, \Omega^p)\,.$$

(6.2) Theorem. *For a smooth oriented n-dimensional manifold X, the transformation P induces an isomorphism*

$$P: H_c^{n-p}(X, \mathbf{C}) \to H_p^c(X, \mathbf{C})\,, \quad p \in \mathbf{N}\,,$$

from de Rham cohomology with compact support to de Rham homology.

Proof. The following diagram is commutative

(6.3) $$\begin{array}{ccc} \Gamma_c(X, \Omega^{n-p}) & \overset{P}{\to} & D_p^c(X, \mathbf{C}) \\ \downarrow (-1)^n d & & \downarrow (-1)^{p-1} b \\ \Gamma_c(X, \Omega^{n-p+1}) & \overset{P}{\to} & D_{p-1}^c(X, \mathbf{C}) \end{array}$$

as it follows from the relation

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{n-p} \alpha \wedge d\beta\,, \quad \alpha \in \Gamma_c(X, \Omega^{n-p}), \beta \in \Gamma(X, \Omega^p)\,,$$

using that $\int d(\alpha \wedge \beta) = 0$. Upon replacing X by an arbitrary open subset we obtain a morphism of complexes of sheaves

(6.4) $$P : \Omega^{\cdot}[n] \to \Omega^{\cdot \vee}$$

with the signs of the differentials modified according to the commutative diagram (6.3). The morphism (6.4) is a quasi-isomorphism as it follows by checking the case $X = \mathbf{R}^n$ by means of the Poincaré lemma with and without compact support. As in the proof of (2.1) we conclude that P induces a quasi-isomorphism

$$P : \Gamma_c(X, \Omega^{\cdot}[n]) \to \Gamma_c(X, \Omega^{\cdot \vee}).$$

The second complex may be identified with $D^c_{\cdot}(X, \mathbf{C})$ as we have seen in (2.3) and the result follows by passing to homology. Q.E.D.

Let us extend Poincaré duality to the relative groups of an open subset U of X with complement Z in X. With the notation of (6.1), the operator P from (6.4) induces a commutative diagram

$$\begin{array}{ccccccc} 0 \to \Gamma_c(U, \Omega^{\cdot}[n]) & \to & \Gamma_c(X, \Omega^{\cdot}[n]) & \to & \Gamma_c(Z, \Omega^{\cdot}[n]) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow P & & \\ 0 \to D^c_{\cdot}(U, \mathbf{C}) & \overset{j_*}{\to} & D^c_{\cdot}(X, \mathbf{C}) & \to & D^c_{\cdot}(X, U; \mathbf{C}) & \to & 0 . \end{array}$$

Again the differentials in the bottom row must be modified as in (6.3). The unmarked vertical arrows are the quasi-isomorphisms of Poincaré duality. The vertical arrow marked P is induced by the algebra of the diagram. Again, from algebra we deduce a *quasi-isomorphism*

(6.5) $$P : \Gamma_c(Z, \Omega^{\cdot}[n]) \to D^c_{\cdot}(X, U; \mathbf{C}), \quad Z = X - U .$$

Passing to homology we obtain the Poincaré duality isomorphism

(6.6) $$P : H^{n-p}_c(Z, \mathbf{C}) \xrightarrow{\sim} H^c_p(X, U; \mathbf{C}) .$$

The p'th sheaf cohomology group with compact support $H^p_c(Z, \mathbf{C})$ has the following de Rham representation

(6.7) $$\{\omega \in \Gamma_c(X, \Omega^p) \mid \operatorname{Supp}(d\omega) \subseteq U\} \Big/ \begin{array}{l} \{d\nu \mid \nu \in \Gamma_c(X, \Omega^{p-1})\} \\ + \{\omega \in \Gamma_c(X, \Omega^p) \mid \operatorname{Supp}(\omega) \subseteq U\} \end{array}$$

as it follows from the exact sequence making up the top row of the diagram above.