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Here $A > 0$ depends on F in an explicit way. The first of these is a very deep result of Deligne [5] (and was called the Ramanujan-Petersson conjecture); the second is a fundamental result of Rankin [14]. Also, the Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

is absolutely convergent if $\operatorname{Re} s > (k+1)/2$ and satisfies the functional equation

$$\xi(s) = (2\pi)^{-s}\Gamma(s)L(s) = (-1)^{k/2}\xi(k-s).$$

The most famous example of the Fourier coefficients of a cusp form is the Ramanujan-tau function defined by the identity

$$\Delta(z) = \prod_{n=1}^{\infty} (1-e^{2\pi i n z})^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}. \qquad (\operatorname{Im} z > 0)$$

The function $\Delta(z)$ is a cusp form for Γ of weight $k = 12$. See Apostol [1] for this example and some of the basic results mentioned above.

In this paper we study the summatory function

$$A(x) = \sum_{n \leqslant x} a(n)$$

and the error term in the formula

$$\int_0^x A^2(t)dt = Cx^{k+1/2} + B(x),$$

where $C > 0$ is given explicitly by

$$C = \frac{1}{(4k+2)\pi^2} \sum_{n=1}^{\infty} \frac{a^2(n)}{n^{k+1/2}}.$$

It is known that

$$B(x) = \begin{cases} O(x^k \log^2 x), \\ \Omega\left(x^{k-1/4} \dfrac{(\log\log\log x)^3}{\log x}\right). \end{cases}$$

The upper bound was obtained by Walfisz [21] (in the special case of the Ramanujan τ-function, but the general case follows the same lines), and was reproved by Chandrasekharan and Narasimhan [3] in greater generality. The omega result may be found in Ivić [6, Theorem 4].

Our first result is a slight improvement over what has probably been in the folklore for years, but does not seem to have appeared in print before.

THEOREM 1. *We have*

$$A(x) = O(x^{(k-1)/2+1/3}).$$

Proof. First, by partial summation and a result found in Perelli [13], we observe that

$$\sum_{p \leqslant x} \frac{|a(p)|^2}{p^k} = \log\log x + O(1).$$

This implies by the Cauchy-Schwarz inequality and the well-known estimate $\sum_{p \leqslant x} 1/p = \log\log x + O(1)$ that

$$\sum_{p \leqslant x} |a(p)|\, p^{(1-k)/2} p^{-1} \leqslant \left\{\sum_{p \leqslant x} \frac{|a(p)|^2}{p^k}\right\}^{1/2} \left\{\sum_{p \leqslant x} \frac{1}{p}\right\}^{1/2} .$$
$$= \log\log x + O(1).$$

Hence by Deligne's estimate and a result of Shiu [20], we have uniformly for $x^\varepsilon \ll y \leqslant x$,

$$\sum_{x<n\leqslant x+y} |a(n)| \ll x^{(k-1)/2} \sum_{x<n\leqslant x+y} |a(n)|\, n^{(1-k)/2}$$
$$\ll \frac{x^{(k-1)/2} y}{\log x} \exp\left\{\sum_{p \leqslant x+y} |a(p)|\, p^{(1-k)/2} p^{-1}\right\}$$
$$\ll x^{(k-1)/2} y. \tag{2}$$

Now all we need to do to complete the proof of the theorem is to apply Chandrasekharan and Narasimhan's general Theorem 4.1 and Remark (5.5) of [2] with $\delta = k, A = 1, q = -\infty, u = 1/4$ and $\eta = 1/6$ to deduce the theorem. We have taken $y = x^{1/2-\eta}$. In the folklore, the result is given with an additional factor of a power of $\log x$ which arises by using a weaker form of Shiu's result. (The first appearance in print of a result of this kind, without proof and with x^ε instead of $\log x$, can be found in [15].)

In correspondence, Rankin has claimed a similar result with a negative power of the logarithm:

$$A(x) = O\left(x^{(k-1)/2+1/3} (\log x)^{-\delta+\varepsilon}\right),$$

where $\delta = (8-3\sqrt{6})/10$. This uses much deeper and more precise information about the distribution of values of $|a(n)|$, including the analytic behavior of the Dirichlet series $\sum_{n=1}^{\infty} a(n)^4 n^{-s}$ due to Moreno and Shahidi [10].

Our next result is a significant improvement over the results of Joris [8] and Redmond [18]. It also gives the natural analogue of the Ω_- results of Corrádi and Kátai [4] for Dirichlet's divisor problem.

THEOREM 2. *There exists a positive constant* D *such that*

$$A(x) = \Omega_{\pm}\left(x^{k/2-1/4} \exp\left\{D \frac{(\log\log x)^{1/4}}{(\log\log\log x)^{3/4}}\right\}\right).$$

Proof. This result is an immediate consequence of the general theorem of Redmond [18] (actually, the Corollary to Theorem 2) and the following result of Murty [11]: for any $\varepsilon > 0$, the inequality

$$|a(p)| > (\sqrt{2}-\varepsilon)p^{(k-1)/2}$$

holds for a positive proportion of primes p. To apply Redmond's result, we let P be this set of primes (with $\varepsilon = .1$, say). Then the hypotheses of the cited Corollary are satisfied and the result follows.

The key to this method is a lower bound for the sum

$$\sum_{q \in Q_x} |a(q)|\, q^{-k/2-1/4},$$

where Q_x is the set of all square-free numbers composed entirely of primes less than or equal to x and in P. Joris obtains only the lower bound $c \log x$. In our first success with this problem, we used the simple inequality

$$\sum_{q \in Q_x} |a(q)|\, q^{-k/2-1/4} \gg \sum_{n \leqslant x} |a(n)|\, n^{-k/2-1/4}$$

and the deeper result of Rankin [17] that the latter sum is bounded below by $cx^{1/4}(\log x)^{2^{-1/2}-1}$. The sum in question however can be written as a finite Euler product, in which case Murty's result gives the much improved lower bound

$$c' \exp\{cx^{1/4}/\log x\}.$$

Finally, we give an improved omega result for $B(x)$.

THEOREM 3. *With* D *given by Theorem 2,*

$$B(x) = \Omega\left(x^{k-1/4} \exp\left\{3D \frac{(\log\log x)^{1/4}}{(\log\log\log x)^{3/4}}\right\}\right).$$

Proof. The method of proof is exactly that used in Ivić-Ouellet [7] where it is applied to the analogous problem for the divisor function. It is also the same method used by Ivić [6] to obtain the Ω-result mentioned in (3) above. In that paper however, he did not have the stronger omega result of Theorem 2. We reconstruct the proof here for completeness.

First note that, uniformly for $x^{\varepsilon} \ll H \leqslant x$,

$$A(x) = H^{-1}\int_x^{x+H} A(t)dt + O(x^{(k-1)/2}H).$$

This can easily be shown using Deligne's and Shiu's bounds already applied above in (2).

For simplicity of notation, put

$$T(x) = \exp\left\{D\frac{(\log\log x)^{1/4}}{(\log\log\log x)^{3/4}}\right\}.$$

Note that $T(x)$ is increasing for x sufficiently large and that $T(2x)/T(x) \to 1$ as x tends to infinity.

Suppose the theorem does not hold so that for any $\varepsilon > 0$ and for $x > x_0(\varepsilon)$,

$$B(x) \leqslant \varepsilon^2 x^{k-1/4} T(x)^3.$$

We may assume that $x_0(\varepsilon)$ is large enough so that $T(x)$ is increasing for $x > x_0(\varepsilon)$.

Now let x be a generic point where the Ω_+-result of Theorem 2 holds. Then by the Cauchy-Schwarz inequality, we have, for $x^{\varepsilon} \ll H \leqslant x$ and some positive constants c_1, c_2, and c_3,

$$\begin{aligned} c_1 x^{k/2-1/4}T(x) &\leqslant H^{-1/2}\left(\int_x^{x+H} A^2(t)dt\right)^{1/2} + c_2 H x^{(k-1)/2} \\ &\leqslant c_3 x^{k/2-1/4} + H^{-1/2}\,|\,B(x+H) - B(x)\,|^{1/2} + c_2 H x^{(k-1)/2} \\ &\leqslant c_3 x^{k/2-1/4} + 2\varepsilon H^{-1/2} x^{k/2-1/8} T(2x)^{3/2} + c_2 H x^{(k-1)/2}. \end{aligned}$$

We put $H = \varepsilon x^{1/4} T(2x)$, divide this last relation by $x^{k/2-1/4}T(x)$ and let x tend to infinity (through the sequence of special values where the Ω_+-result of Theorem 2 holds). In so doing we deduce

$$c_1 \leqslant 2\varepsilon^{1/2} + c_2\varepsilon.$$

Since, by assumption, ε could be arbitrarily small, this contradiction proves the theorem.

It seems reasonable to conjecture that

$$B(x) \ll x^{k-1/4+\varepsilon}.$$

If true, this conjecture is very strong since it implies (by the method of proof given above) the classical conjecture

$$A(x) \ll x^{k/2-1/4+\varepsilon}.$$

In conclusion we make a few remarks on extending these results to Maass wave forms. For more background on Maass wave forms and their role in analytic number theory, see the article by Iwaniec in [16]. A Maass wave form $f(z)$ is an automorphic function for Γ, i.e., a function of weight zero which is an eigenfunction for the hyperbolic Laplacian. It has a Fourier series expansion (in $x = \operatorname{Re} z$) of the form

$$f(z) = \sum_{n=1}^{\infty} a_n \sqrt{y} K_{ir}(2\pi ny) \begin{cases} \cos(2\pi nx) & \text{if } f \text{ is even} \\ \sin(2\pi nx) & \text{if } f \text{ is odd}, \end{cases}$$

where K_{ir} the K-Bessel function and r is related to f's Laplacian eigenvalue. Note that these forms are not holomorphic, only real-analytic.

One usually assumes that f is also an eigenfunction for the Hecke operators (as we did above for holomorphic forms) so that, properly normalized, the coefficients a_n are multiplicative and real. The associated Dirichlet series satisfies a (slightly more complicated) functional equation (see Maass [9]) so the theorems of Chandrasekharan-Narasimhan [2] and Redmond [18] still apply. Rankin's result (1) carries over to this case easily. (One takes $k = 1$ in that result and elsewhere because of a slightly odd normalization.) However, the Ramanujan-Petersson conjecture $|a_n| \leqslant d(n)$ has not been settled for these coefficients. The most that has been shown is

$$(3) \qquad |a_n| \ll n^{1/5+\varepsilon}$$

and

$$|a_p| > 1.189$$

for a positive proportion of primes p. The first was observed independently by a number of people and follows from the work of Shahidi [19]. The second is due to Murty. See Murty's paper [12] for a proof of both.

Consequently, Theorem 2 above holds for these coefficients *mutatis mutandis* but Theorems 1 and 3 do not as they depend on the inequality (2) which requires the Ramanujan-Petersson conjecture. Using Rankin's result (1) or inequality (3), one can of course give a weaker estimate for the sum