

# 7. WINDING NUMBERS

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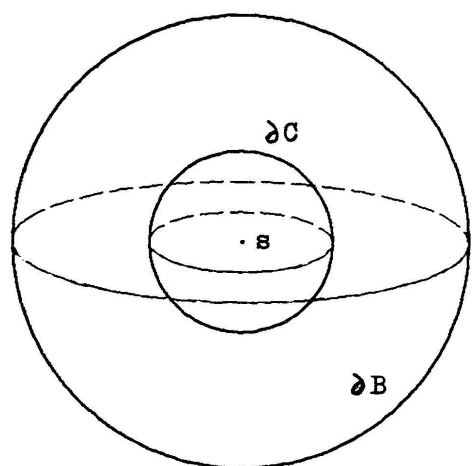
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## 7. WINDING NUMBERS

Let  $X$  be an  $n$ -dimensional oriented smooth manifold and  $s$  a point of  $X$ . Consider a compact  $n$ -dimensional submanifold with boundary  $B$  with  $s$  as an interior point and put

$$(7.1) \quad \text{Tr}(\omega; s) = \int_{\partial B} \omega, \quad \omega \in \Gamma(X - \{s\}, \Omega^{n-1}), d\omega = 0.$$

This symbol is independent of  $B$  as it follows by considering a small "ball"  $C$  around  $s$  contained in the interior of  $B$



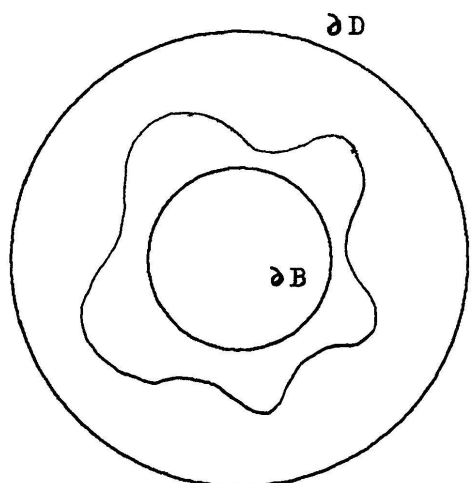
Stokes formula for  $B - C^0$

$$\int_{\partial B} \omega - \int_{\partial C} \omega = \int_{B - C^0} d\omega.$$

Alternatively, choose a compactly supported smooth real function  $\rho_s$  on  $X$  which is constant 1 in a neighbourhood of  $s$ . Then

$$(7.2) \quad \text{Tr}(\omega; s) = (-1)^n \int_X \omega \wedge d\rho_s, \quad \omega \in \Gamma(X - \{s\}, \Omega^{n-1}), d\omega = 0.$$

*Proof.* Choose "balls"  $B$  and  $D$  with center  $s$  such that  $\rho_s$  is constant 1 on  $B$  while  $\text{Supp}(\rho_s)$  is contained in the interior of  $D$ . From Stokes formula we get that



$$\begin{aligned} \int_{\partial D} \rho_s \omega - \int_{\partial B} \rho_s \omega &= \int_{D-B} \rho_s \wedge \omega - \int_{D-B} \rho_s \omega \\ &- \int_{\partial B} \omega = -(-1)^n \int_X \omega \wedge d\rho_s + \int_{D-B} \rho_s d\omega. \end{aligned}$$

Notice that the last terms vanishes in case  $\omega$  is exact.

Q.E.D.

(7.3) *Example.* Let  $E$  denote an oriented  $n$ -dimensional Euclidian space. The distance  $r$  to the origin defines a 1-form  $dr^{2-n}$  on  $E - \{0\}$ . The dual form  $*dr^{2-n}$  in the sense of Hodge is closed with

$$\text{Tr}(*dr^{2-n}; 0) = (2-n)\sigma_{n-1}$$

where  $\sigma_{n-1}$  denotes the area of the unit sphere in  $E$ , compare [3] VII. 1.

Let us interpret (7.1) in terms of de Rham homology. Integration of  $n$ -forms on  $X$  over the manifold  $B$  determines a compact  $n$ -chain on  $X$  whose boundary, as written in (7.1), has support in  $X - \{s\}$ . The corresponding relative homology class

$$(7.4) \quad \theta_s \in H_n^c(X, X - \{s\}; \mathbf{C}), \quad s \in X,$$

is independent of  $B$ : with the notation above, the compact  $n$ -chain  $\int_B - \int_C$  has support in  $X - \{s\}$ . The relative homology class we have just constructed is often called *the local orientation class*.

(7.5) PROPOSITION. *Let  $s$  be a point of an oriented  $n$ -dimensional smooth manifold  $X$ . The local orientation class  $\theta_s$  generates  $H_n^c(X, X - \{s\}; \mathbf{C})$ .*

*Proof.* With the terminology from section 5 we may express formula (7.1) by means of the local orientation class

$$(7.6) \quad \text{Tr}(\omega; s) = \langle \theta_s, \partial\omega \rangle = \langle b\theta_s, \omega \rangle, \quad \omega \in H^{n-1}(X - \{s\}, \mathbf{C}).$$

In case  $n > 2$  we conclude from (7.3), that  $\theta_s \neq 0$ . The case  $n = 2$  is left with the reader.

Q.E.D.

Let us remark that formula (7.2) shows how to identify  $\theta_s$  under relative Poincaré duality (6.6).

(7.7) PROPOSITION. *Let  $S$  be a finite subset of the oriented  $n$ -dimensional compact manifold  $X$ . For any closed form  $\omega \in \Gamma(X - S, \Omega^n)$  we have that*

$$\sum_{s \in S} \text{Tr}(\omega; s) = 0.$$

*Proof.* Let the *fundamental class*  $\theta \in H_n(X, \mathbf{C})$  be given by

$$\langle \theta, \omega \rangle = \int_X \omega, \quad \omega \in \Gamma(X, \Omega^n).$$

Let us consider a point  $s \in S$  and use the notation from (7.1). The difference  $\int_X - \int_B$  has support in  $X - \{s\}$ , which shows that the image of  $\theta$  in  $H_n(X, X - \{s\}; \mathbf{C})$  is  $\theta_s$ . We have that

$$\sum_{s \in S} \text{Tr}(\omega; s) = \sum_{s \in S} \langle \theta_s, \partial\omega \rangle = \langle \theta_S, \partial\omega \rangle = \langle b\theta_S, \omega \rangle$$

where  $\theta_s$  denotes the restriction of  $\theta$  to  $H_n(X, X - S; \mathbf{C})$ . Conclusion by the fact that  $b\theta_S = 0$ . Q.E.D.

(7.8) *Definition.* Let  $\gamma$  be a compact  $n$ -chain on the oriented  $n$ -dimensional smooth manifold  $X$ . For a point  $s \in X$  outside  $\text{Supp}(b\gamma)$  the class of  $\gamma$  in  $H_n^c(X, X - \{s\}; \mathbf{C})$  can be written

$$[\gamma] = \text{Ind}(\gamma; s)\theta_s, \quad \text{Ind}(\gamma; s) \in \mathbf{C}.$$

The number  $\text{Ind}(\gamma; s)$  is called the *winding number* of  $\gamma$  with respect to  $s$ .

(7.9) *Example.* Let  $K$  denote an  $n$ -dimensional compact submanifold with boundary. Integration over  $K$  defines a compact  $n$ -chain  $\kappa$  with  $\text{Supp}(\partial\kappa) = \partial K$ . The winding number for  $\kappa$  is 1 in the interior of  $K$  and 0 outside  $K$ .

(7.10) **THEOREM.** *Let  $\gamma$  be a compact  $n$ -chain on the oriented  $n$ -dimensional smooth manifold  $X$ . The winding number  $s \mapsto \text{Ind}(\gamma; s)$  is a locally constant function on the complement of  $\text{Supp}(b\gamma)$  in  $X$ . This function is zero outside some compact subset of  $X$  containing  $\text{Supp}(b\gamma)$ .*

*Proof.* Let us consider an arbitrary open subset  $U$  of  $X$  containing  $\text{Supp}(b\gamma)$ . We shall now use relative Poincaré duality to describe the class of  $\gamma$  in  $H_n^c(X, U; \mathbf{C})$ . According to (6.6) and (6.7) we can represent  $\gamma$  by a relative  $n$ -chain of the form

$$\langle \gamma, v \rangle = \int_X \rho v, \quad v \in \Gamma(X, \Omega^n)$$

where  $\rho$  is a compactly supported smooth function on  $X$ , constant in a neighbourhood of any point  $s$  of  $Z = X - U$ . Let us notice that

$$\langle \partial\gamma, \omega \rangle = (-1)^n \int \omega \wedge d\rho, \quad \omega \in \Gamma(U, \Omega^{n-1}), \quad d\omega = 0.$$

In order to calculate  $\text{Ind}(\gamma; s)$  we replace  $U$  by a small pointed neighbourhood  $D^*$  of  $s$ . With the notation of (7.2) let us write  $\rho = \rho(s)\rho_s$  and deduce that

$$\langle \partial\gamma, \omega \rangle = \rho(s)\text{Tr}(\omega; s), \quad \omega \in \Gamma(D^*, \Omega^{n-1}), \quad d\omega = 0.$$

We can now conclude from (7.6) that

$$\text{Ind}(\gamma; s) = \rho(s), \quad s \in X - U.$$

This reveals that  $s \mapsto \text{Ind}(\gamma; s)$  is a compactly supported, locally constant function on  $X - U$ .

For a given fixed point  $s \notin \text{Supp}(b\gamma)$  choose  $U$  to be an open neighbourhood of  $\text{Supp}(b\gamma)$  with  $\bar{U}$  compact and  $s \notin U$ . We can apply the considerations above and conclude that the winding number is constant in a neighbourhood of  $s$  and zero outside some compact neighbourhood of  $\text{Supp}(b\gamma)$ . Q.E.D.

(7.11) COROLLARY. *Let  $\gamma$  be a compact  $n$ -chain on the oriented smooth manifold  $X$  and  $U$  an open subset of  $X$  containing  $\text{Supp}(b\gamma)$ . The relative de Rham homology class*

$$[\gamma] \in H_n^c(X, U; \mathbb{C})$$

is zero if and only if  $\text{Ind}(\gamma; s) = 0$  for all  $s \in X - U$ .

*Proof.* This is a corollary to the proof of (7.10) rather than the statement (7.10). Anyway, the basic point is Poincaré duality (6.6). Q.E.D.

### 8. CAUCHY'S RESIDUE THEOREM

We shall consider a smooth map  $\gamma: S^{n-1} \rightarrow E$  from the oriented  $n - 1$  sphere into an oriented  $n$ -dimensional real vector space  $E$ . For a point  $s$  outside  $\gamma(S^{n-1})$  pick a closed  $(n - 1)$ -form  $\omega_s$  on  $E - \{s\}$  with  $\text{Tr}(\omega_s; s) = 1$  and define the *winding number* of  $\gamma$  with respect to  $s$  to be

$$(8.1) \quad \text{Ind}(\gamma; s) = \int_{S^{n-1}} \gamma^*\omega_s.$$

(8.2) CAUCHY'S RESIDUE THEOREM. *Let  $\gamma: S^{n-1} \rightarrow X$  denote a smooth map into an open subset  $X$  of  $E$  with  $\text{Ind}(\gamma; z) = 0$  for all  $z \in E - X$ .*