

**Zeitschrift:** L'Enseignement Mathématique  
**Band:** 35 (1989)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** CAUCHY RESIDUES AND DE RHAM HOMOLOGY  
**Kapitel:** 8. Cauchy's residue theorem  
**Autor:** Iversen, Birger  
**DOI:** <https://doi.org/10.5169/seals-57358>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

**Download PDF:** 07.10.2024

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

In order to calculate  $\text{Ind}(\gamma; s)$  we replace  $U$  by a small pointed neighbourhood  $D^*$  of  $s$ . With the notation of (7.2) let us write  $\rho = \rho(s)\rho_s$  and deduce that

$$\langle \partial\gamma, \omega \rangle = \rho(s)\text{Tr}(\omega; s), \quad \omega \in \Gamma(D^*, \Omega^{n-1}), \quad d\omega = 0.$$

We can now conclude from (7.6) that

$$\text{Ind}(\gamma; s) = \rho(s), \quad s \in X - U.$$

This reveals that  $s \mapsto \text{Ind}(\gamma; s)$  is a compactly supported, locally constant function on  $X - U$ .

For a given fixed point  $s \notin \text{Supp}(b\gamma)$  choose  $U$  to be an open neighbourhood of  $\text{Supp}(b\gamma)$  with  $\bar{U}$  compact and  $s \notin U$ . We can apply the considerations above and conclude that the winding number is constant in a neighbourhood of  $s$  and zero outside some compact neighbourhood of  $\text{Supp}(b\gamma)$ . Q.E.D.

(7.11) COROLLARY. *Let  $\gamma$  be a compact  $n$ -chain on the oriented smooth manifold  $X$  and  $U$  an open subset of  $X$  containing  $\text{Supp}(b\gamma)$ . The relative de Rham homology class*

$$[\gamma] \in H_n^c(X, U; \mathbb{C})$$

is zero if and only if  $\text{Ind}(\gamma; s) = 0$  for all  $s \in X - U$ .

*Proof.* This is a corollary to the proof of (7.10) rather than the statement (7.10). Anyway, the basic point is Poincaré duality (6.6). Q.E.D.

### 8. CAUCHY'S RESIDUE THEOREM

We shall consider a smooth map  $\gamma: S^{n-1} \rightarrow E$  from the oriented  $n - 1$  sphere into an oriented  $n$ -dimensional real vector space  $E$ . For a point  $s$  outside  $\gamma(S^{n-1})$  pick a closed  $(n - 1)$ -form  $\omega_s$  on  $E - \{s\}$  with  $\text{Tr}(\omega_s; s) = 1$  and define the *winding number* of  $\gamma$  with respect to  $s$  to be

$$(8.1) \quad \text{Ind}(\gamma; s) = \int_{S^{n-1}} \gamma^* \omega_s.$$

(8.2) CAUCHY'S RESIDUE THEOREM. *Let  $\gamma: S^{n-1} \rightarrow X$  denote a smooth map into an open subset  $X$  of  $E$  with  $\text{Ind}(\gamma; z) = 0$  for all  $z \in E - X$ .*

For a closed and discrete subset  $S$  of  $X$  disjoint from  $\gamma(S^{n-1})$  only finitely many of the numbers  $\text{Ind}(\gamma; s), s \in S$ , are distinct from zero and

$$\int_{S^{n-1}} \gamma^* \omega = \sum_{s \in S} \text{Ind}(\gamma; s) \text{Tr}(\omega; s)$$

for any closed form  $\omega$  on  $X - S$ .

*Proof.* The long exact de Rham homology sequence for the pair  $X - S, E$  degenerates into an isomorphism

$$b: H_n^c(E, X - S; \mathbf{C}) \xrightarrow{\sim} H_{n-1}^c(X - S, \mathbf{C}).$$

Let us view  $\gamma$  as a homology class on  $X - S$  and introduce the class

$$b^{-1}\gamma \in H_n^c(E, X - S; \mathbf{C}).$$

Let us notice that the winding number (8.1) and (7.8) agree. Thus we conclude from (7.11) that  $b^{-1}\gamma$  maps to zero in  $H_n^c(E, X; \mathbf{C})$  and consequently that  $\gamma$  is homologous to zero on  $X$ . The exact sequence

$$0 \rightarrow H_n^c(X, X - S; \mathbf{C}) \xrightarrow{b} H_{n-1}^c(X - S, \mathbf{C}) \rightarrow H_{n-1}^c(X, \mathbf{C})$$

allows us to interpret  $\gamma$  as a relative class

$$\gamma \in H_n^c(X, X - S; \mathbf{C}).$$

The class  $\gamma$  can be specified by the formula

$$\langle b\gamma, \omega \rangle = \int_{S^{n-1}} \gamma^* \omega, \quad \omega \in \Gamma(X - S, \Omega^{n-1}), \quad d\omega = 0.$$

From the decomposition (4.9) and excision (4.6) we deduce a canonical isomorphism

$$H_n(X, X - S; \mathbf{C}) \xrightarrow{\sim} \bigoplus_{s \in S} H_n(X, X - \{s\}; \mathbf{C})$$

which allow us to decompose the class  $\gamma$  into a finite sum, compare (7.6),

$$\gamma = \sum_{s \in S} \text{Ind}(\gamma; s) \theta_s.$$

Using the general Stokes formula (5.3) we get that

$$\langle b\gamma, \omega \rangle = \langle \gamma, \partial\omega \rangle = \sum \text{Ind}(\gamma; s) \langle \theta_s, \partial\omega \rangle$$

and the result follows from formula (7.6).

Q.E.D.