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HURWITZ-RADON MATRICES AND PERIODICITY MODULO 8

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0. INTRODUCTION

0.1. We consider complex $n \times n$ — matrices $A_1, A_2, ..., A_s$, either all unitary (case U) or all orthogonal (case O); they are called Hurwitz-Radon matrices, in short HR-matrices, if

(1) $\quad A_j^2 = -E, \; A_jA_k + A_kA_j = 0, \quad j, k = 1, 2, ..., s, \; j \neq k;$

E or E_n denotes the unit matrix. Such matrices are well-known to exist, even with entries $0, \pm 1, \pm i$ (case U) or $0, \pm 1$ (case O). The possible values of n are multiples $mn_0, m = 1, 2, 3, ...$ where in case $U, n_0 = 2^{s/2}$ if s is even, $n_0 = 2^{(s-1)/2}$ if s is odd. In case $O, n_0 = 2^{(s-1)/2}$ if $s \equiv 7 \bmod 8$; $n_0 = 2^{s/2}$ if $s \equiv 0, 6$; $n_0 = 2^{(s+1)/2}$ if $s \equiv 1, 3, 5$; and $n_0 = 2^{(s+2)/2}$ if $s \equiv 2, 4 \bmod 8$.

If we put $A_0 = E$ the relations (1) are equivalent to

$$f_s(x_0, x_1, ..., x_s) = \sum_0^s x_jA_j$$

being a unitary, or orthogonal respectively, matrix for all real x_j with $\sum_0^s x_j^2 = 1$. Thus f_s can be considered as a map $S^s \to U$ via $U(n)$, or $S^s \to O$ via $O(n)$ where $U = \varinjlim U(k)$ and $O = \varinjlim O(k)$ are the infinite unitary and orthogonal groups. We also write f_s for the homotopy class of that map, $f_s \in \pi_s(U)$ or $\pi_s(O)$. We recall that by the Bott periodicity theorems these groups are cyclic or 0.

THEOREM A. *If $A_1, A_2, ..., A_s$ are HR-matrices of minimal size $n = n_0(s)$ then f_s is a generator of $\pi_s(U)$, or $\pi_s(O)$ respectively, $s = 0, 1, 2, ...$.*

Remark 1. For $s = 0$ (empty set of HR-matrices) we have $f_0(x_0) = x_0(1)$, $x_0^2 = 1$; i.e., $f_0(1) = (1)$, $f_0(-1) = (-1)$, $f_0 : S^0 \to O(1) \to O$. For $s > 0$, $f_0 : S^s \to O$ clearly factors through $SO(n) \to SO$ (U being connected, the analogue is irrelevant in the unitary case).

Remark 2. The problem originally solved by Hurwitz [H] and Radon [R] refers to the case O: One asks for complex bilinear forms $z = f(x, y) = (\sum_0^s x_j A_j) y$, where $z = (z_1, ..., z_n)$, $y = (y_1, ..., y_n)$, $x = (x_0, ..., x_s)$, such that

$$z_1^2 + ... + z_n^2 = (x_0^2 + ... + x_s^2)(y_1^2 + ... + y_n^2).$$

This means that $\sum_0^s x_j A_j$ is orthogonal, i.e. leaves invariant $\sum_0^n y_j^2$ except for the factor $\sum_0^s x_j^2$; and thus, since we may assume $A_0 = E$, that $A_1, ..., A_s$ is a set of orthogonal HR-matrices in the sense of (1).

The case U refers to the analogous problem for the identity

$$|z_1|^2 + ... + |z_n|^2 = (x_0^2 + ... + x_s^2)(|y_1|^2 + ... + |y_n|^2)$$

where y and z are complex, and x real.

0.2. The symplectic case: It is also of interest to consider HR-matrices, i.e. matrices fulfilling (1), which are symplectic. A linear combination $\sum_0^s x_j A_j$ of $2n \times 2n$-matrices with $A_0 = E$ is symplectic up to the factor $\sum_0^s x_j^2$ if and only if $A_1, ..., A_s$ is a set of symplectic HR-matrices (Proposition 4.1).

We restrict attention to unitary symplectic matrices, i.e., to the group $Sp(n) \subset U(2n)$, and write Sp for the infinite symplectic group $\lim\limits_{\rightarrow} Sp(k)$. With a set $A_1, ..., A_s$ of unitary symplectic HR-matrices, and $A_0 = E$, we associate the map $f_s(x_0, x_1, ..., x_s) = \sum_0^s x_j A_j$, $\sum_0^s x_j^2 = 1$, of S^s into Sp via $Sp(n)$; we also write f_s for the corresponding element of $\pi_s(Sp)$, known to be 0 or cyclic.

THEOREM A'. *If* $A_1, ..., A_s$ *are unitary symplectic HR-matrices of minimal size* $2n_0$ *then* f_s *is a generator of* $\pi_s(Sp)$.

0.3. The paper is organized as follows. We first recall (Section 1) that the HR-matrix problem can be formulated in terms of representations of certain finite group G_s, $s = 0, 1, 2, ...$ introduced by the author [E], and discuss these representations using the elegant description of [LS]. In Section 2 the "reduced" Grothendieck groups of representations E_s^U and E_s^O are computed; they turn out to be isomorphic to $\pi_s(U)$ and $\pi_s(O)$ respectively. Moreover a product is defined in the direct sum of the $E_s^U(E_s^O)$ turning it into a graded ring $E_*^U(E_*^O)$. The claim of Theorem A is proved in Section 3; we show that the maps $\phi: E_s^U \to \pi_s(U)$, $\psi: E_s^O \to \pi_s(O)$ given by the f_s of 0.1 are isomorphisms. Using the product structure in $\pi_*(U)$ and $\pi_*(O)$ known from K-theory the proof reduces to simple verifications in low dimensions. The symplectic case is dealt with in Section 4. In Section 5 we make a remark concerning the "linearization phenomenon" for the homotopy groups of U, O and Sp.

1. The groups G_s and their representations

1.1. We will denote throughout by G_s the group given by the presentation

$$G_s = <\varepsilon, a_1, ..., a_s \mid \varepsilon^2 = 1,\ a_j^2 = \varepsilon,\ a_j a_k = \varepsilon a_k a_j,\ j, k = 1, 2, ..., s, j \neq k> .$$

Clearly any set $A_1, ..., A_s$ of HR-matrices yields a (unitary or orthogonal) representation of G_s of degree n by $\varepsilon \mapsto -E$, $a_j \mapsto A_j$, $j = 1, 2, ..., s$. Conversely a representation of G_s with $\varepsilon \mapsto -E$, in short an ε-representation, yields a set of s HR-matrices. For the elementary properties of G_s and its representations we refer to [E]. We just recall that the order of G_s is 2^{s+1}, that ε is central, and that the irreducible unitary ε-representations of G_s are of degree $2^{s/2}$ if s is even (one equivalence class), of degree $2^{(s-1)/2}$ if s is odd (two equivalence classes). These degrees are the minimal values n_0 in case U. As for the case O, one has to recall that a representation is equivalent to an orthogonal one if and only if it is equivalent to a real (and orthogonal) one. Thus, unless an irreducible unitary ε-representation is already real, one has to add its conjugate-complex representation, and the discussion of the various cases depending on s yields the minimal values n_0 (case O) mentioned in the introduction; in other words, the degrees of the irreducible orthogonal ε-representations of G_s.

1.2. A very simple and useful scheme for studying the groups G_s and their ε-representations (and many other things) has been deviced by T. Y. Lam and T. Smith [LS]. They have expressed the G_s as products of very small and well-known groups. Namely $C = G_1$, the cyclic group of order 4; $Q = G_2$, the quaternionic group of order 8; K, the Klein 4-group; and D, the dihedral group of order 8. Although K and D do not belong to the family G_s, they are of a similar nature and contain a distinguished central element ε of order 2 (distinguished arbitrarily in K). "Product" here means the central product obtained from the direct product by identifying the two ε's. The expression for the G_s then is as follows, displaying a fundamental periodicity modulo 8:

(2)

s	0	1	2	3	4	5	6	7	8	9	...
G_s	$\mathbf{Z}/2$	C	Q	QK	QD	D^2C	D^3	D^3K	D^4	D^4C	...

and $G_{s+8} = D^4 G_s$.

The tensor product of ε-representations of two of the groups G_s, K, D is an ε-representation of their product above, and all ε-representations of the G_s can be obtained in that explicit way from those of C, Q, K, D, which are well-known. This yields, in particular, the characters χ and the Schur indices I of the irreducible unitary ε-representation (the Schur index $I = 1$ if the representation is equivalent to a real one; if it is not, $I = -1$ if it is equivalent to the conjugate-complex one, $I = 0$ otherwise). Both χ and I behave multiplicatively with respect to the central product.

1.3. The Schur indices of the irreducible ε-representations are: 0 for $C = G_1$, -1 for $Q = G_2$, and 1 for K and D (two equivalence classes for K, one for D). This yields the Schur indices I_s of the irreducible ε-representations of the G_s, as listed in (2) below; we further list the numbers ν_s^U of inequivalent unitary, and ν_s^O of inequivalent orthogonal irreducible ε-representations, and the respective degrees d_s^U, d_s^O. Note that I_s is periodic with period 8, and d_s^O is periodic with period 8 in the sense that $d_{s+8}^O = 16 d_s^O$. Finally we include in the same table the Grothendieck groups D_s^U and D_s^O of (equivalence classes of) irreducible ε-representations of G_s, with respect to the direct sum of representations.

(3)

s	0	1	2	3	4	5	6	7	8	9	...
I_s	1	0	−1	−1	−1	0	1	1	1	0	...
ν_s^U	1	2	1	2	1	2	1	2	1	2	
ν_s^O	1	1	1	2	1	1	1	2	1	1	
d_s^U	1	1	2	2	4	4	8	8	16	16	
d_s^O	1	2	4	4	8	8	8	8	16	32	
D_s^U	$\mathbf{Z}$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}\oplus\mathbf{Z}$	
D_s^O	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	

The values of d_s^O follow immediately from the I_s and the d_s^U. The values n_0 for the case O, as given in the Introduction, are the d_s^O.

2. THE REDUCED ε-REPRESENTATION RING

2.1. For all $s \geqslant 0$ the group G_s is the subgroup of G_{s+1} obtained by omitting the generator a_{s+1}; let $h_s \colon G_s \to G_{s+1}$ be the embedding homomorphism. Via h_s we can restrict an ε-representation of G_{s+1} to G_s, which in terms of HR-matrices means omitting A_{s+1}.

Let $h_s^* \colon D_{s+1}^U \to D_s^U$ be the corresponding homomorphism of Grothendieck groups, and $E_s^U = D_s^U / h_s^* D_{s+1}^U$ the "reduced" groups; similarly $E_s^O = D_s^O / h_s^* D_{s+1}^O$. They can easily be computed by means of the characters of ε-representations, as follows.

For Q and D the character of an irreducible unitary ε-representation is 0 except on 1 and ε. For C and K it is $\neq 0$ on all 4 elements; on the essential generator ($\neq$ε) of C it is $+i$ or $-i$ for the two inequivalent representations, and $+1$ or -1 in the case of K. For G_s, s even, we infer from the table (2) that the character is 0 except on $1, \varepsilon$. For G_s, s odd, the character is 0 except on $1, \varepsilon$ and two further elements $z, \varepsilon z$; on these the two inequivalent ε-representations differ just by the sign of the character.

If s is even, $d_{s+1}^U = d_s^U = 2^{s/2}$; thus the restriction of an irreducible ε-representation must be irreducible, whence $h_s^* D_{s+1}^U = D_s^U$, $E_s^U = 0$. If s is odd, $d_{s+1}^U = 2d_s^U = 2^{(s+1)/2}$; thus the restriction is the sum of two irreducible ε-representations, and since the character is 0 (except on $1, \varepsilon$) these two must be inequivalent. Therefore $h_s^* D_{s+1}^U$ is the "diagonal" of $D_s^U = \mathbf{Z} \oplus \mathbf{Z}$, and $E_s^U = \mathbf{Z}$; its generator ρ_s is represented by either of the two inequivalent irreducible ε-representations of G_s, $-\rho_s$ by the other one.

In the orthogonal case the E_s^O are computed similarly from (3). Since $d_1^O = 2$ and $d_0^O = 1$, the restriction from D_1^O to D_0^O yields twice the generator, and $E_0^O = \mathbf{Z}/2$; the same argument holds for $s \equiv 0 \bmod 8$, $d_{s+1}^O = 2d_s^O$. Since $d_2^O = 4$ and $d_1^O = 2$, we get $E_1^O = \mathbf{Z}/2$. From $d_3^O = d_2^O = 4$ we get $E_2^O = 0$. As for $s = 3$, the character argument shows that $h_3^* D_4^O =$ diagonal of $D_3^O (= \mathbf{Z} \oplus \mathbf{Z})$, and $E_3^O = \mathbf{Z}$. For $s = 4, 5, 6$ the dimensions $d_{s+1}^O = d_s^O$ show that $E_4^O = E_5^O = E_6^O = 0$. For $s = 7$, the character argument yields $h_7^* D_8^O =$ diagonal of $D_7^O (= \mathbf{Z} \oplus \mathbf{Z})$, and $E_7^O = \mathbf{Z}$. Finally one has, for all s, $E_{s+8}^O \cong E_s^O$.

These results are summarized in the table

(4)

s	0	1	2	3	4	5	6	7	8	9	...
E_s^U	0	**Z**	0	**Z**	0	**Z**	0	**Z**	0	**Z**	
E_s^O	**Z**/2	**Z**/2	0	**Z**	0	0	0	**Z**	**Z**/2	**Z**/2	

According to the Bott periodicity theorems the above table is just that of the $\pi_s(U)$ and $\pi_s(O)$, $s = 0, 1, 2, \ldots$. Before studying the relation as stated in Theorem A we establish product structures in the reduced Grothendieck groups of ε-representations, i.e., of HR-matrices.

2.2. We consider HR-matrices $A_1, A_2, \ldots, A_s \in U(n)$ and put, for

$$x = (x_0, x_1, \ldots, x_s) \in \mathbf{R}^{s+1}$$

and $A_0 = E_n$ ($n \times n$ unit matrix)

$$f(x) = \sum_0^s x_j A_j .$$

For all x with $|x| = 1$, $f(x)$ is a unitary matrix: this is, as mentioned in the Introduction, precisely the meaning of the HR-matrix relations (1).

Let further $B_1, B_2, \ldots, B_t \in U(m)$ be HR-matrices, and for

$$y = (y_0, y_1, \ldots, y_t) \in \mathbf{R}^{t+1}, \quad B_0 = E_m,$$

$$g(y) = \sum_0^t y_k B_k ;$$

$g(y) \in U(m)$ for all y with $|y| = 1$. We define F by

$$F(x, y) = \begin{pmatrix} f(x) \otimes E_m & E_n \otimes g(y) \\ - E_n \otimes \overline{g(y)}^T & \overline{f(x)}^T \otimes E_m \end{pmatrix} .$$

One immediately checks that $F(x, y)\bar{F}^T(x, y) = (|x|^2 + |y|^2)E_{2nm}$. Thus $F(x, y) \in U(2nm)$ for all $(x, y) \in \mathbf{R}^{s+t+2}$ with $|x|^2 + |y|^2 = 1$. Since the coefficient matrix of x_0 is E_{2nm} the coefficient matrices of $x_1, \ldots, x_s, y_0, \ldots, y_t$ constitute a set of $s + t + 1$ HR-matrices $\in U(2nm)$. They are, explicitly,

$$(5) \quad \begin{pmatrix} A_j \otimes E_m & 0 \\ 0 & - A_j \otimes E_m \end{pmatrix}, \begin{pmatrix} 0 & E_{nm} \\ - E_{nm} & 0 \end{pmatrix}, \begin{pmatrix} 0 & E_n \otimes B_k \\ E_n \otimes B_k & 0 \end{pmatrix}$$

with $j = 1, \ldots, s$ and $k = 1, \ldots, t$. In other words, we have a product of ε-representations of G_s and G_t

$$D_s^U \times D_t^U \overset{\cup}{\to} D_{s+t+1}^U .$$

Since addition in D_s^U is by the direct sum of ε-representations this product is clearly distributive. Associativity (up to equivalence) is easily checked. We thus get a ring structure in $D_*^U = \bigoplus_{-1}^{\infty} D_s^U$; we have added the term $D_{-1}^U = \mathbf{Z}$ generated by the ring unit. The ring D_*^U is graded if the grading is by $s + 1$ for D_s.

From the HR-matrices (5) of the product one notes that if one of the two factors is restricted from D_*^U so is the product; i.e., $h*D_*^U$ is a (graded) ideal in D_*^U, and we get a (graded) ring structure in $D_*^U/h*D_*^U = E_*^U$.

The same procedure yields, of course, a (graded) ring structure in $E_*^O = \bigoplus_{s=-1}^{\infty} E_s^O$, with grading $s + 1$ for E_s^O. In 2.3 and 2.4 below these rings are described explicitly.

Remark 2.1. An easy computation shows that the rings E_*^U and E_*^O are anticommutative with respect to the grading, i.e., commutative except for the factor $(-1)^{(s+1)(t+1)}$. This will not really be used since the E_s^U and E_s^O are all 0, $\mathbf{Z}$ or $\mathbf{Z}/2$. We just note that in the case $\mathbf{Z}$, with generator ρ_s, $-\rho_s$ is given by the other equivalence class of irreducible ε-representations, see 2.1.

2.3. The ring E_*^U.

The generator ρ_s of E_s^U, given by an irreducible unitary ε-representation of G_s, has degree $2^{s/2}$ if s is even, $2^{(s-1)/2}$ if s is odd. The product $\rho_s\rho_t \in E_{s+t+1}^U$ has degree

$$\begin{aligned} &2^{(s+t+2)/2} && \text{if} \quad s \text{ and } t \text{ are even},\\ &2^{(s+t+1)/2} && \text{if} \quad s \text{ is even, } t \text{ odd, or vice-versa},\\ &2^{(s+t)/2} && \text{if} \quad s \text{ and } t \text{ are odd}. \end{aligned}$$

Thus, unless both s and t are even, the product is irreducible, i.e., $\rho_s\rho_t = \pm\rho_{s+t+1}$. After choice of $\rho_1 \in E_1^U$ we can choose $\rho_3 = \rho_1^2$, $\rho_5 = \rho_1\rho_3 = \rho_3\rho_1 = \rho_1^3, \ldots$, and for all odd $s = 2r - 1$, $\rho_s = \rho_1^r$; for even s, $E_s^U = 0$.

PROPOSITION 2.2. *The product with* $\rho_1 \in E_1^U$ *is an isomorphism* $E_s^U \cong E_{s+2}^U$ *for all* s. *For odd* $s = 2l - 1$ *we choose*

$$\rho_{2l-1} = \rho_1^l, \; l = 1, 2, 3, \ldots .$$

THEOREM 2.3. *E_*^U is the polynomial ring* $\mathbf{Z}[\rho_1]$.

2.4. THE RING E_*^O.

We denote by σ_s the generator of E_s^O (= 0 if $s \equiv 2, 4, 5, 6$ modulo 8; determined up to sign if $s \equiv 3, 7$ modulo 8 where $E_s^O = \mathbf{Z}$).

The generator ρ_7 $(= \rho_1^4) \in E_7^U$ can be given by a real ε-representation of degree 8 which we can use as generator $\sigma_7 \in E_7^O$. The ring homomorphism $\Phi: E_*^O \to E_*^U$ induced by the embedding $O \to U$, $\Phi(\sigma_7) = \rho_7$, is thus an isomorphism $E_7^O \cong E_7^U$. In E_*^O the degree of $\sigma_7\sigma_s \in E_{s+8}^O$ is $16d_s^O = d_{s+8}^O$. Hence $\sigma_7\sigma_s$ is irreducible, i.e., $= \pm \sigma_{s+8}$ for all s. In particular we can choose $\sigma_{15} = \sigma_7^2, \sigma_{23} = \sigma_7^3, \dots, \sigma_{8r-1} = \sigma_7^r$.

PROPOSITION 2.4. *The isomorphism $E_s^O \cong E_{s+8}^O$ can be given by the product with $\sigma_7 \in E_7^O$.*

PROPOSITION 2.5. *$\sigma_7 \in E_7^O$ generates a subring of E_*^O which is the polynomial ring* $\mathbf{Z}[\sigma_7]$.

We further note that $\sigma_3 \in E_3^O$ is mapped by Φ to $2\rho_3 \in E_3^U$. From $\Phi(\sigma_3^2) = 4\rho_3^2 = 4\rho_7 = \Phi(4\sigma_7)$ we infer that $\sigma_3^2 = 4\sigma_7$. As for $\sigma_0 \in E_0^O$, it is of degree 1 and order 2, and $\sigma_0^2 \in E_1^O$ is of degree 2 and order 2, i.e., $\sigma_0^2 = \sigma_1$. Of course $\sigma_0^3 = 0$.

In summary:

THEOREM 2.6. *E_*^O is the commutative ring, graded by $s+1$ for E_s^O, generated by $\sigma_0, \sigma_3, \sigma_7$ with the only relations* $2\sigma_0 = 0$, $\sigma_0^3 = 0$, $\sigma_3^2 = 4\sigma_7$.

3. THE HOMOTOPY GROUPS OF U AND O

3.1. We will deal explicitly with the unitary case. The orthogonal case can be treated in almost exactly the same way; any additional arguments will be mentioned wherever necessary.

In the Introduction 0.1 we associated with a set of s unitary $n \times n$ HR-matrices, i.e., with an ε-representation of G_s, a map $f: S^s \to U$ of the s-sphere $S^s \subset \mathbf{R}^{s+1}$ into the infinite unitary group U via $U(n)$. Since conjugation is homotopic to the identity, equivalent representations yield homotopic maps f (in the orthogonal case, we have to observe that conjugation can be made with a matrix from the identity component). The map $\phi: D_s^U \to \pi_s(U)$ thus obtained is a homomorphism; indeed, homotopy group addition of f and f' in $\pi_s(U(n))$ can be replaced by multiplication in

$U(n)$; this is homotopic in $U(2n)$ to the map $\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}$, and on the other hand addition in D_s^U is defined through the direct sum of representations.

If the ε-representation is restricted from D_{s+1}^U, i.e., if the set of HR-matrices belongs to a set of $s+1$ HR-matrices, f extends to a map $S^{s+1} \to U$ and is thus nullhomotopic. The homomorphism ϕ therefore induces a homomorphism $E_s^U \to \pi_s(U)$, again written ϕ. The analogue $E_s^O \to \pi_s(O)$ will be denoted by ψ. The groups E_s^U and E_s^O are 0 or cyclic generated by irreducible ε-representations, i.e., by HR-matrices of minimal size. Our claim, Theorem A, can therefore be reformulated as follows.

THEOREM B. *The homomorphisms* $\phi : E_s^U \to \pi_s(U)$ *and* $\psi : E_s^O \to \pi_s(O)$ *are isomorphisms,* $s = 0, 1, 2, \ldots$.

3.2. For small values of s the claim is easily checked.

Case U

$s = 1$: E_1^U can be generated by one HR-matrix $A_1 = (i)$. Thus

$$f(x_0, x_1) = (x_0 + ix_1) \in U(1)$$

if $x_0^2 + x_1^2 = 1$. This is a generator of $\pi_1(U(1)) \cong \pi_1(U) = \mathbf{Z}$.

$s = 3$: E_3^U is generated by 3 HR-matrices

$$A_1 = \begin{pmatrix} i & \\ & -i \end{pmatrix}, \quad A_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad A_3 = \begin{pmatrix} & i \\ i & \end{pmatrix}.$$

Thus

$$f(x_0, x_1, x_2, x_3) = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} \in SU(2)$$

if $\sum_0^3 x_j^2 = 1$. This is a generator of $\pi_3(SU(2))\,[= \pi_3(S^3)] \cong \pi_3(U) = \mathbf{Z}$.

Case O

$s = 0$: Empty set of HR-matrices, $f(x_0) = (x_0) \in O(1)$ if $x_0^2 = 1$, $x_0 = \pm 1$. This is a generator of $\pi_0(O(1)) \cong \pi_0(O) = \mathbf{Z}/2$.

$s = 1$: E_1^O is generated by one HR-matrix $A_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. Thus

$$f(x_0, x_1) = \begin{pmatrix} x_0 & x_1 \\ -x_1 & x_0 \end{pmatrix} \in SO(2)$$

if $x_0^2 + x_1^2 = 1$. This is a generator of $\pi_1(SO(2)) = \mathbf{Z}$; as a map $S^1 \to SO(3)$ it is a generator of $\pi_1(SO(3)) \cong \pi_1(O) = \mathbf{Z}/2$.

$s = 3$: E_3^O is generated by three 4×4 HR-matrices which yield

$$f(x_0, x_1, x_2, x_3) = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ -x_1 & x_0 & x_3 & -x_2 \\ -x_2 & -x_3 & x_0 & x_1 \\ -x_3 & x_2 & -x_1 & x_0 \end{pmatrix} \in SO(4)$$

if $\sum_0^3 x_j^2 = 1$. This is a map $S^3 \to SO(4)$ which is well-known to become, under $SO(4) \to SO(5)$, a generator of $\pi_3(SO(5)) \cong \pi_3(O) = \mathbf{Z}$.

3.3. The proof of Theorem B becomes very simple if ϕ and ψ are turned into ring homomorphisms $E_*^U \to \pi_*(U) = \bigoplus_{-1}^{\infty} \pi_s(U)$ ($\pi_{-1} = \mathbf{Z}$ generated by the ring unit) and $E_*^O \to \pi_*(O)$. For this purpose we have to define a product in $\pi_*(U)$ and $\pi_*(O)$, graded by $s + 1$ for π_s. This is done by extending the product introduced in 2.2 from linear maps $f: S^s \to U$ or O to arbitrary continuous maps.

Given a continuous map $f: S^s \to U$ via $U(n)$,

$$S^s = \{x = (x_0, x_1, \ldots, x_s) \in \mathbf{R}^{s+1} \quad \text{with} \quad |x| = 1\},$$

we extend it to $f_0: \mathbf{R}^{s+1} \to M_n(\mathbf{C})$ by $f_0(x) = |x| f\left(\frac{x}{|x|}\right)$, $f_0(0) = 0$. Similarly for $g: S^t \to U$ via $U(m)$, $S^t = \{y \in \mathbf{R}^{t+1}$ with $|y| = 1\}$. Then

$$F(x, y) = \begin{pmatrix} f_0(x) \otimes E_m & E_n \otimes g_0(y) \\ -E_n \otimes \overline{g_0(y)}^T & \overline{f_0(x)}^T \otimes E_m \end{pmatrix}$$

is a unitary $2nm \times 2nm$ matrix for all $(x, y) \in \mathbf{R}^{s+t+2}$ with $|x|^2 + |y|^2 = 1$ and thus defines a map $F: S^{s+t+1} \to U$ via $U(2nm)$. Homotopic maps f, or g respectively, yield homotopic F and we obtain a product $F = f \cup g$

$$\pi_s(U) \times \pi_t(U) \xrightarrow{\cup} \pi_{s+t+1}(U).$$

From the description of homotopy group addition in $\pi_s(U)$ as given above in 3.1 one easily checks that $f \cup g$ is distributive. Thus $\pi_*(U)$ is a ring, and so is $\pi_*(O)$, graded by $s + 1$ for $\pi_s(U)$ or $\pi_s(O)$.

3.4. Bott periodicity is usually expressed in terms of complex and real K-theory. We thus use the isomorphisms

$$\pi_s(U) \cong \tilde{K}_{\mathbf{C}}(S^{s+1}) \quad \text{and} \quad \pi_s(O) \cong \tilde{K}_{\mathbf{R}}(S^{s+1}).$$

We recall that $\pi_s(U) \cong \tilde{K}_{\mathbf{C}}(S^{s+1})$ is obtained through $\pi_s(U) \cong K_{\mathbf{C}}(B^{s+1}, S^s$ where B^{s+1} is the unit ball $\{x \in \mathbf{R}^{s+1}, |x| \leqslant 1\}$; the element correspondin to $f \in \pi_s(U)$ is given by two (trivial) **C**-vector bundles over B^{s+1}, identifie on S^s by means of f. It will not come as a surprise that $f \cup g$ abov corresponds to the $\cup$-product

$$K_{\mathbf{C}}(B^{s+1}, S^s) \times K_{\mathbf{C}}(B^{t+1}, S^t) \to K_{\mathbf{C}}(B^{s+t+2}, S^{s+t+1})$$

given by the external tensor product of bundles. Indeed the map $f \cup g$ $= F: S^{s+t+1} \to U$ via $U(2nm)$ can be interpreted as follows: One decom poses $S^{s+t+1} \subset \mathbf{R}^{s+t+2}$ (coordinates $x_0, x_1, \ldots, x_s, y_0, y_1, \ldots, y_t$ with $|x|^2$ $+ |y|^2 = 1$) into $\{|x|^2 \leqslant \frac{1}{2}, |y|^2 \geqslant \frac{1}{2}\}$ homeomorphic to $B^{s+1} \times S^t$ an $\{|x|^2 \geqslant \frac{1}{2}, |y|^2 \leqslant \frac{1}{2}\}$ homeomorphic to $S^s \times B^{t+1}$; the map F is

$$\begin{pmatrix} f(x) \otimes E_m & 0 \\ 0 & \overline{f(x)}^T \otimes E_m \end{pmatrix} \quad \text{on} \quad S^s \times (0), \text{ i.e. } y = 0, |x| = 1,$$

$$\begin{pmatrix} 0 & E_n \otimes g(y) \\ -E_n \otimes \overline{g(y)}^T & 0 \end{pmatrix} \quad \text{on} \quad (0) \times S^t, \text{ i.e. } x = 0, |y| = 1.$$

Under $K_{\mathbf{C}}(B^{s+1}, S^s) \cong \tilde{K}_{\mathbf{C}}(S^{s+1})$ one then has a graded ring structure i $\bigoplus_{-1}^{\infty} \tilde{K}_{\mathbf{C}}(S^{s+1})$ isomorphic to $\pi_*(U)$. According to the Bott periodicity theore (see [K], p. 123) this ring is the polynomial ring $\mathbf{Z}[a]$ generated by th generator of $\tilde{K}_{\mathbf{C}}(S^2)$; i.e., $\pi_*(U)$ is the polynomial ring generated by th generator a of $\pi_1(U)$.

Similarly, $\pi_*(O)$ is the commutative ring with generators $b_0 \in \pi_0(O$ $b_3 \in \pi_3(O)$, $b_7 \in \pi_7(O)$ with relations $2b_0 = 0$, $b_0^3 = 0$, $b_3^2 = 4b_7$ ([K p. 156-157).

To prove Theorem B we therefore only have to show:

Case U. $\rho_1 \in E_1^U$ is mapped by ϕ to $a \in \pi_1(U)$.

Case O. $\sigma_0 \in E_0^O$ is mapped by ψ to $b_0 \in \pi_0(O)$ and $\sigma_3 \in E_3^O$ to $b_3 \in \pi_3(O$

This has already been done in 3.2.

4. Symplectic HR-matrices

4.1. Symplectic matrices A leave invariant the bilinear form with coefficient matrix $J = \begin{pmatrix} & E_n \\ -E_n & \end{pmatrix}$; i.e., $A^T J A = J$. With respect to the HR-matrix relations (1) they behave exactly like orthogonal or unitary matrices:

Proposition 4.1. *Let* $A_1, A_2, \ldots, A_s$ *be* $2n \times 2n$*-matrices, and* $A_0 = E_{2n}$. *Then* $\sum_0^s x_j A_j$ *is symplectic up to the factor* $\sum_0^1 x_j^2$ *for all* $x_0, x_1, \ldots, x_s$ *if and only if* $A_1, A_2, \ldots, A_s$ *is a set of symplectic HR-matrices.*

Proof.

$$\left(\sum_0^s x_j A_j^T\right) J \left(\sum_0^s x_j A_j\right) = \sum_0^s x_j^2 A_j^T J A_j$$

$$+ \sum_1^s x_0 x_j (A_j^T J + J A_j) + \sum_{j,k=1}^s x_j x_k (A_j^T J A_k + A_k^T J A_j), \quad j \neq k.$$

Assume $A_j^T J A_j = J, j = 0, \ldots, s$; and

$$A_j^2 = -E, A_j A_k + A_k A_j = 0, \; j, k = 1, \ldots, s, j \neq k.$$

Then $-A_j^T J = J A_j$, and $A_j^T J A_k + A_k^T J A_j = -J(A_j A_k + A_k A_j) = 0$. Thus the whole expression reduces to $\left(\sum_0^s x_j^2\right) J$. The argument is plainly reversible.

4.2. In the following, "symplectic" will mean unitary symplectic; i.e., we consider matrices from the compact group $Sp(n) \subset U(2n)$. A set of symplectic HR-matrices $A_1, A_2, \ldots, A_s$ is thus an ε-representation of G_s in $Sp(n)$; we continue to call its degree $2n$. The notations ν_s^{Sp}, d_s^{Sp}, D_s^{Sp}, E_s^{Sp} have the same meaning as before for U and for O.

All elements of G_s have square 1 or ε; a matrix $\in U(2n)$ of square $\pm E$ is symplectic if and only if it is of the form $\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$ with $B^t = -B$, $\bar{A}^T = A$ in the case of square E, and $B^t = B$, $\bar{A}^t = -A$ in the case of square $-E$. Symplectic representations of G_s are sums of irreducible unitary representations; if an irreducible unitary ε-representation is not (equivalent to *a*) symplectic, we have to add its conjugate-complex in order to obtain an irreducible symplectic ε-representation. Due to the description (2) of the G_s the following observations yield the complete list of degrees etc.

4.3. (a) The tensor product of a unitary representation V of even degree and an orthogonal representation (of any degree) is symplectic if and only if V is.

(b) Since $Sp(1) = SU(2)$, the irreducible unitary ε-representations (of degree 2) of $G_2 = Q$ are symplectic.

(c) The irreducible ε-representations of D (= dihedral group of order 8) are not symplectic, but orthogonal; the same holds for D^j and D^jK, K = Klein 4-group.

(d) The tensor product of any representation with the irreducible ε-representation (of degree 1) of $G_1 = C$ is not symplectic.

The periodicity modulo 8, $G_{s+8} = G_8 G_s = D^4 G_s$, with $d_8^O = d_8^U = 16$, yields $d_{s+8}^{Sp} = 16 d_s^{Sp}$ and $\nu_{s+8}^{Sp} = \nu_s^{Sp}$. For $s \equiv 2, 3, 4$ modulo 8 the irreducible unitary ε-representations of G_s are symplectic, $d_s^{Sp} = d_s^U$ and $\nu_s^{Sp} = \nu_s^U$; for the other s they are not, thus $d_s^{Sp} = 2d_s^U$. For $s \equiv 1, 5$ modulo 8 the conjugate-complex representations are inequivalent, thus $\nu_s^{Sp} = 1$; for $s \equiv 0, 6, 7$ we combine two equivalent representations, thus $\nu_s^{Sp} = \nu_s^U$, i.e., $\nu_s^{Sp} = 1$ for $s \equiv 0, 6$ and $\nu_s^{Sp} = 2$ for $s \equiv 7$. The restriction arguments from G_{s+1} to G_s are as before and yield the E_s^{Sp}, which are periodic modulo 8.

We summarize the results in the following table

(6)

s	0	1	2	3	4	5	6	7	8	9
ν_s^{Sp}	1	1	1	2	1	1	1	2	1	1
d_s^{Sp}	2	2	2	2	4	8	16	16	32	32
D_s^{Sp}	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$
E_s^{Sp}	0	0	0	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	$\mathbf{Z}$	0	0

4.4. Comparing with (3) one notes that $D_s^O \cong D_{s+4}^{Sp}$ and $E_s^O \cong E_{s+4}^{Sp}$. The isomorphisms can be made explicit in terms of the $\cup$-product introduced in 2.2, as follows.

Let $\rho_3 \in D_3^U = D_3^{Sp}$ be one of the generators, $\rho_3 = \bar{\rho}_3$, and $\sigma_t \in D_t^O$ one of the generators. The product $\rho_3 \cup \sigma_t \in D_{t+4}^U$ has degree $2.2.d_t^O$; this is precisely the degree of a generator of D_{t+4}^{Sp}. We check that $\rho_3 \cup \sigma_t$ is indeed in D_{t+4}^{Sp} and thus a generator: this is clear for $t \equiv 0, 6, 7$, $t + 4 \equiv 2, 3, 4$ modulo 8 where $D_{t+4}^{Sp} = D_{t+4}^U$; for $t \equiv 1, 2, 3, 4, 5$ we know that $\sigma_t = \rho_t + \bar{\rho}_t$, whence $\rho_3 \cup \sigma_t = \rho_3 \cup \rho_t + \bar{\rho}_3 \cup \bar{\rho}_t$, i.e., it is one of the generators of D_{t+4}^{Sp}.

THEOREM 4.1. *The product of the generator* $\rho_3 \in E_3^U = E_3^{Sp}$ *with* E_s^O *is an isomorphism* $E_s^O \cong E_{s+4}^{Sp}$ *for all* $s \geqslant 0$.

4.5. We now consider the homomorphism $\theta \colon E_s^{Sp} \to \pi_s(Sp)$, analogous to ϕ and ψ before.

Let $A_1, A_2, ..., A_s$ be a set of s symplectic $2n \times 2n$ HR-matrices, and $A_0 = E$. Then

$$f_s(x_0, x_1, ..., x_s) = \sum_0^s x_j A_j$$

$x = (x_0, x_1, ..., x_s) \in \mathbf{R}^{s+1}, \sum_0^s x_j^2 = 1$, is symplectic. We consider f_s as a map $S^s \to Sp$ via $Sp(n)$; as in the cases U and O this yields a homomorphism $\theta \colon E_s^{Sp} \to \pi_s(Sp)$, $s \geqslant 0$. The $\pi_s(Sp)$ are known to be 0 or cyclic. Theorem A′ can now be reformulated as follows.

THEOREM B′. θ *is an isomorphism* $E_s^{Sp} \to \pi_s(Sp)$, $s \geqslant 0$.

For $s = 3$ this is clear: since $E_3^{Sp} = E_3^U$ and $\pi_3(Sp) \cong \pi_3(Sp(1)) = \pi_3(SU(2)) \cong \pi_3(U)$, $c = \theta(\rho_3)$ is a generator of $\pi_3(Sp) = \mathbf{Z}$.

To complete the proof of Theorem B′ we use, as for Theorem B, the $\cup$-product and results of K-theory relating $K_{\mathbf{R}}$ with $K_{\mathbf{H}}$, the quaternionic or symplectic K-theory. The product $c \cup b$, $b \in \pi_s(O)$, can be expressed in terms of linear maps $S^3 \to Sp(1) = SU(2)$, $S^s \to O(m)$, $S^{s+4} \to U(4m)$. As seen in 4.3, it lies in fact in $Sp(2m) \subset U(4m)$ and can thus be regarded as an element of $\pi_{s+4}(Sp)$. The map $c \cup - \colon \pi_s(O) \to \pi_{s+4}(Sp)$ corresponds, under $\pi_s(O) \cong \tilde{K}_{\mathbf{R}}(S^{s+1})$ and $\pi_t(Sp) \cong \tilde{K}_{\mathbf{H}}(S^{t+1})$, to the isomorphism $\tilde{K}_{\mathbf{R}}(S^{s+1}) \to \tilde{K}_{\mathbf{H}}(S^{s+5})$ given by the external tensor product of bundles with the generating bundles of $\tilde{K}_{\mathbf{H}}(S^4) = \mathbf{Z}$ (see [K], p. 154). Hence $c \cup -$ is an isomorphism $\pi_s(O) \cong \pi_{s+4}(Sp)$.

Moreover, since everything is described by linear maps the diagram

$$\begin{array}{ccc} E_s^O & \xrightarrow{\psi} & \pi_s(O) \\ \rho_3 \cup - \downarrow & & \downarrow c \cup - \\ E_{s+4}^{Sp} & \xrightarrow{\theta} & \pi_{s+4}(Sp) \end{array}$$

is commutative. The upper and the two vertical maps being isomorphisms, so is θ.

5. Linearization

5.1. The groups E_s^U can be viewed, through the homomorphism $\phi: E_s^U \to \pi_s(U)$ in 3.1, as "linear homotopy groups" of U. This means that we consider maps of S^s into U via some $U(n)$ which are linear in the coordinates $x_0, x_1, ..., x_s$ of $\mathbf{R}^{s+1} \supset S^s$; and linear nullhomotopies, i.e., extensions to $S^{s+1} \to U(n)$ linear in $x_0, x_1, ..., x_{s+1}$. It is an immediate corollary of Theorem B that these linear homotopy groups $\pi_s^{\mathrm{lin}}(U)$ are isomorphic to the $\pi_s(U)$ by the obvious imbedding $\pi_s^{\mathrm{lin}}(U) \to \pi_s(U)$. In other words:

Any map $S^s \to U$ is homotopic to a linear map, and if a linear map $S^s \to U$ is nullhomotopic then it admits a linear nullhomotopy.

Similar statements hold, of course, for $\pi_s(O)$ and $\pi_s(Sp)$.

5.2. If these linearization phenomena could be established directly (by some approximation procedure) one would obtain a very transparent proof of the Bott periodicity theorems for $\pi_s(U)$, $\pi_s(O)$, and $\pi_s(Sp)$, in the sense that they would be reduced to the algebraic computation of E_s^U, E_s^O, and E_s^{Sp} as carried out here.

5.3. Linear maps $S^s \to U$ via $U(n)$, etc., are given explicitly in terms of HR-matrices; thus the coefficients involve 0, $\pm$ 1, $\pm$ i only. Such maps have a meaning over very general fields instead of $\mathbf{R}$ and $\mathbf{C}$, and one should compare the corresponding linear homotopy groups with homotopy groups defined by means of algebraic maps.

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