4. Symplectic HR-matrices

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4. Symplectic HR-matrices

4.1. Symplectic matrices A leave invariant the bilinear form with coefficient matrix $J = \begin{pmatrix} & E_n \\ -E_n & \end{pmatrix}$; i.e., $A^T J A = J$. With respect to the HR-matrix relations (1) they behave exactly like orthogonal or unitary matrices:

Proposition 4.1. *Let* $A_1, A_2, \ldots, A_s$ *be* $2n \times 2n$*-matrices, and* $A_0 = E_{2n}$. *Then* $\sum_0^s x_j A_j$ *is symplectic up to the factor* $\sum_0^1 x_j^2$ *for all* $x_0, x_1, \ldots, x_s$ *if and only if* $A_1, A_2, \ldots, A_s$ *is a set of symplectic HR-matrices.*

Proof.
$$\left(\sum_0^s x_j A_j^T\right) J \left(\sum_0^s x_j A_j\right) = \sum_0^s x_j^2 A_j^T J A_j$$
$$+ \sum_1^s x_0 x_j (A_j^T J + J A_j) + \sum_{j,k=1}^s x_j x_k (A_j^T J A_k + A_k^T J A_j), \quad j \neq k.$$

Assume $A_j^T J A_j = J, j = 0, \ldots, s$; and
$$A_j^2 = -E, A_j A_k + A_k A_j = 0, \ j, k = 1, \ldots, s, j \neq k.$$
Then $-A_j^T J = J A_j$, and $A_j^T J A_k + A_k^T J A_j = -J(A_j A_k + A_k A_j) = 0$. Thus the whole expression reduces to $\left(\sum_0^s x_j^2\right) J$. The argument is plainly reversible.

4.2. In the following, "symplectic" will mean unitary symplectic; i.e., we consider matrices from the compact group $Sp(n) \subset U(2n)$. A set of symplectic HR-matrices $A_1, A_2, \ldots, A_s$ is thus an ε-representation of G_s in $Sp(n)$; we continue to call its degree $2n$. The notations ν_s^{Sp}, d_s^{Sp}, D_s^{Sp}, E_s^{Sp} have the same meaning as before for U and for O.

All elements of G_s have square 1 or ε; a matrix $\in U(2n)$ of square $\pm E$ is symplectic if and only if it is of the form $\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$ with $B^t = -B$, $\bar{A}^T = A$ in the case of square E, and $B^t = B$, $\bar{A}^t = -A$ in the case of square $-E$. Symplectic representations of G_s are sums of irreducible unitary representations; if an irreducible unitary ε-representation is not (equivalent to *a*) symplectic, we have to add its conjugate-complex in order to obtain an irreducible symplectic ε-representation. Due to the description (2) of the G_s the following observations yield the complete list of degrees etc.

4.3. (a) The tensor product of a unitary representation V of even degree and an orthogonal representation (of any degree) is symplectic if and only if V is.

(b) Since $Sp(1) = SU(2)$, the irreducible unitary ε-representations (of degree 2) of $G_2 = Q$ are symplectic.

(c) The irreducible ε-representations of D (= dihedral group of order 8) are not symplectic, but orthogonal; the same holds for D^j and D^jK, K = Klein 4-group.

(d) The tensor product of any representation with the irreducible ε-representation (of degree 1) of $G_1 = C$ is not symplectic.

The periodicity modulo 8, $G_{s+8} = G_8 G_s = D^4 G_s$, with $d_8^O = d_8^U = 16$, yields $d_{s+8}^{Sp} = 16 d_s^{Sp}$ and $\nu_{s+8}^{Sp} = \nu_s^{Sp}$. For $s \equiv 2, 3, 4$ modulo 8 the irreducible unitary ε-representations of G_s are symplectic, $d_s^{Sp} = d_s^U$ and $\nu_s^{Sp} = \nu_s^U$; for the other s they are not, thus $d_s^{Sp} = 2d_s^U$. For $s \equiv 1, 5$ modulo 8 the conjugate-complex representations are inequivalent, thus $\nu_s^{Sp} = 1$; for $s \equiv 0, 6, 7$ we combine two equivalent representations, thus $\nu_s^{Sp} = \nu_s^U$, i.e., $\nu_s^{Sp} = 1$ for $s \equiv 0, 6$ and $\nu_s^{Sp} = 2$ for $s \equiv 7$. The restriction arguments from G_{s+1} to G_s are as before and yield the E_s^{Sp}, which are periodic modulo 8.

We summarize the results in the following table

(6)

s	0	1	2	3	4	5	6	7	8	9
ν_s^{Sp}	1	1	1	2	1	1	1	2	1	1
d_s^{Sp}	2	2	2	2	4	8	16	16	32	32
D_s^{Sp}	**Z**	**Z**	**Z**	**Z**⊕**Z**	**Z**	**Z**	**Z**	**Z**⊕**Z**	**Z**	**Z**
E_s^{Sp}	0	0	0	**Z**	**Z**/2	**Z**/2	0	**Z**	0	0

4.4. Comparing with (3) one notes that $D_s^O \cong D_{s+4}^{Sp}$ and $E_s^O \cong E_{s+4}^{Sp}$. The isomorphisms can be made explicit in terms of the ∪-product introduced in 2.2, as follows.

Let $\rho_3 \in D_3^U = D_3^{Sp}$ be one of the generators, $\rho_3 = \bar{\rho}_3$, and $\sigma_t \in D_t^O$ one of the generators. The product $\rho_3 \cup \sigma_t \in D_{t+4}^U$ has degree $2.2.d_t^O$; this is precisely the degree of a generator of D_{t+4}^{Sp}. We check that $\rho_3 \cup \sigma_t$ is indeed in D_{t+4}^{Sp} and thus a generator: this is clear for $t \equiv 0, 6, 7$, $t + 4 \equiv 2, 3, 4$ modulo 8 where $D_{t+4}^{Sp} = D_{t+4}^U$; for $t \equiv 1, 2, 3, 4, 5$ we know that $\sigma_t = \rho_t + \bar{\rho}_t$, whence $\rho_3 \cup \sigma_t = \rho_3 \cup \rho_t + \bar{\rho}_3 \cup \bar{\rho}_t$, i.e., it is one of the generators of D_{t+4}^{Sp}.

THEOREM 4.1. *The product of the generator* $\rho_3 \in E_3^U = E_3^{Sp}$ *with* E_s^O *is an isomorphism* $E_s^O \cong E_{s+4}^{Sp}$ *for all* $s \geqslant 0$.

4.5. We now consider the homomorphism $\theta\colon E_s^{Sp} \to \pi_s(Sp)$, analogous to ϕ and ψ before.

Let $A_1, A_2, \ldots, A_s$ be a set of s symplectic $2n \times 2n$ HR-matrices, and $A_0 = E$. Then

$$f_s(x_0, x_1, \ldots, x_s) = \sum_0^s x_j A_j$$

$x = (x_0, x_1, \ldots, x_s) \in \mathbf{R}^{s+1}, \sum_0^s x_j^2 = 1$, is symplectic. We consider f_s as a map $S^s \to Sp$ via $Sp(n)$; as in the cases U and O this yields a homomorphism $\theta\colon E_s^{Sp} \to \pi_s(Sp)$, $s \geqslant 0$. The $\pi_s(Sp)$ are known to be 0 or cyclic. Theorem A′ can now be reformulated as follows.

THEOREM B′. θ *is an isomorphism* $E_s^{Sp} \to \pi_s(Sp)$, $s \geqslant 0$.

For $s = 3$ this is clear: since $E_3^{Sp} = E_3^U$ and $\pi_3(Sp) \cong \pi_3(Sp(1)) = \pi_3(SU(2)) \cong \pi_3(U)$, $c = \theta(\rho_3)$ is a generator of $\pi_3(Sp) = \mathbf{Z}$.

To complete the proof of Theorem B′ we use, as for Theorem B, the $\cup$-product and results of K-theory relating $K_{\mathbf{R}}$ with $K_{\mathbf{H}}$, the quaternionic or symplectic K-theory. The product $c \cup b$, $b \in \pi_s(O)$, can be expressed in terms of linear maps $S^3 \to Sp(1) = SU(2)$, $S^s \to O(m)$, $S^{s+4} \to U(4m)$. As seen in 4.3, it lies in fact in $Sp(2m) \subset U(4m)$ and can thus be regarded as an element of $\pi_{s+4}(Sp)$. The map $c \cup - : \pi_s(O) \to \pi_{s+4}(Sp)$ corresponds, under $\pi_s(O) \cong \tilde{K}_{\mathbf{R}}(S^{s+1})$ and $\pi_t(Sp) \cong \tilde{K}_{\mathbf{H}}(S^{t+1})$, to the isomorphism $\tilde{K}_{\mathbf{R}}(S^{s+1}) \to \tilde{K}_{\mathbf{H}}(S^{s+5})$ given by the external tensor product of bundles with the generating bundles of $\tilde{K}_{\mathbf{H}}(S^4) = \mathbf{Z}$ (see [K], p. 154). Hence $c \cup -$ is an isomorphism $\pi_s(O) \cong \pi_{s+4}(Sp)$.

Moreover, since everything is described by linear maps the diagram

$$\begin{array}{ccc} E_s^O & \xrightarrow{\psi} & \pi_s(O) \\ \rho_3 \cup - \downarrow & & \downarrow c \cup - \\ E_{s+4}^{Sp} & \xrightarrow{\theta} & \pi_{s+4}(Sp) \end{array}$$

is commutative. The upper and the two vertical maps being isomorphisms, so is θ.