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Artikel: THE FIXED POINT SET OF A FINITE GROUP ACTION ON A HOMOLOGY FOUR SPHERE

Kapitel: 5. LOCALLY LINEAR REPRESENTATION

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$\mathrm{Fix}(A_4)$ is a sphere. It cannot be S^2 since the representation of A_4 in $SO(3)$ is irreducible, so it is S^1. The only closed 1-dimensional submanifold of S^1 is S^1 itself, so $\mathrm{Fix}(G) = S^1$.

b. As in subcase a., a linear change in coordinates allows us to assume that h is actually $\tilde{i}$, and as before if $G_2 \in G$ the proposition is proved applying 4.1.

If it is not the case, let α correspond to the cycle $(12345) \in A_5$, β to (123) and γ to (345). We observe that β and γ generate A_5 and so:

1. $\mathrm{Fix}(A_5) = \mathrm{Fix}(\beta) \cap \mathrm{Fix}(\gamma)$,
2. $\mathrm{Fix}(A_5) \subset \mathrm{Fix}(\alpha)$.

We claim that $\mathrm{Fix}(\alpha)$ is S^0. According to Smith's theorem it is enough to prove that the representation of α around x_0 has an isolated fixed point, i.e. is the sum of two irreducible complex ones.

If not by Lemma 3.3 $(\bar{i}(\alpha); i(\alpha))$ would be conjugate in $SO(3) \times SO(3)$ to an element on the diagonal. From the explicit description of i *and* $\bar{i}$ (see the end of section 7.1 of [22]), it follows that they send all the five cycles to non conjugate elements in $SO(3)$, so this is impossible, and $\mathrm{Fix}(\alpha) = S^0$.

As for β and γ, their images under $(\bar{i}, i)$ are conjugate to elements on the diagonal, by 3.3 and 3.4 their fixed point sets have two-dimensional components, and so by Smith's theorem they are copies of S^2.

So $\mathrm{Fix}(G)$ is the intersection of a couple of S^2s and is contained in $\mathrm{Fix}(\alpha)$ which is S^0. If this set is empty or equal to S^0, the proposition follows. If it were a single point, it would be a transverse intersection, by local linearity, but it is not possible since a homology S^4 does not contain any two cycles with intersection number odd. This ends the proof.

5. Locally linear representation

Let's now consider the case of G acting on a homology S^4 with two fixed points, P_0 and P_1.

THEOREM 5.1. *The unoriented representations of* G *around* P_0 *and* P_1 *are linearly equivalent.* [1])

Proof. It will suffice to show that the characters associated to the representations around the P_is agree on every cyclic subgroup C_k of G.

[1]) See the note in the introduction.

Observe that by Lemma 3.4 and Smith's theorem the fixed point set of an element of G different from the identity is either S^0 or S^2.

Let g generate C_k, we distinguish three cases:

1. $\operatorname{Fix}(g^r) = \{P_1; P_2\}$ for every $r \equiv 0 (\operatorname{mod} k)$,
2. $\operatorname{Fix}(g) = S^2$,
3. $\operatorname{Fix}(g) = \{P_1; P_2\}$ but $\operatorname{Fix}(g^n) = S^2$ for some $g^n \neq id$.

Case 1. The hypothesis means that the action is semifree and the claim follows from the work of Atiyah and Bott, see [1] and [14].

Case 2. The action of C_k on the normal bundle of the fixed S^2 defines an element N of $K_{C_k}(S^2)$. Since C_k acts trivially on S^2 the two inclusions $P_i \to S^2$ are obviously C_k homotopic so that the diagram:

$$\begin{array}{ccccc} & \nearrow & K_{C_k}(P_2) & \searrow & \\ [N] \in K_{C_k}(S^2) & & & & R(C_k) \\ & \searrow & K_{C_k}(P_1) & \nearrow & \end{array}$$

commutes. This means that the representation of C_k in the normal component to S^2 are conjugate, the tangential representations are of course both the identity, so the statement is proved.

Case 3. We can assume, by [8], that the action on $S^2 = \operatorname{Fix}(g^n)$ is linear. S^2 has zero intersection number in Σ so its normal bundle N can be identified to $S^2 \times R^2$, and we fix a trivialization. Denote a point of $S^2 - \{P_1; P_2\}$ by (x, t) with $x \in S^1$ and $t \in (0, 1)$. Let C_0 be the space $\{\phi: S^1 \to SO(2) \mid \deg \phi = 0\}$, it is an abelian group by pointwise multiplication and a C_k module with structure given by:

$$(h\phi)(x) = \phi(hx), h \in C_k \quad \text{and} \quad x \in S^1 \subset S^2$$

acted on by the obvious induced action.

By [5], chapter VI, prop. 11.1, the action is given by a θ_t such that

1. $\theta_t \in Z^1(C_k; C_0)$ and depends continuously on $t \in [0, 1]$.
2. $\theta_i(h)(x)$ is constant on $x \in S^1$ and equal to the representation of h at P_i for $i = 0; 1$.

A change in the trivialization adds to each θ_t a coboundary so there is a well defined continuous family $\theta_t: [0, 1] \to H^1(C_k; C_0)$.

A straightforward calculation shows that $H^1(C_k; C_0) = H^2(C_k; Z) = C_k$. Since θ_t is continuous it has to be constant, so $\theta_0 = \theta_1$ and by 2. the two normal representations are equal. In the topological case, by the results of Cappel and Shaneson topological equivalence of matrices in dimension 4 implies linear equivalence, so the statement of Theorem 5.1 makes sense also for a group of homeomorphism.

The proof given can be adapted to this more general case provided that the followings are true:

1. the topological Atiyah-Singer signature formula holds,
2. a locally flat S^2 in Σ has a normal bundle,
3. the argument in case 3 works with Homeo (S^1) instead of $SO(2)$.

Assertion 1 is proved, in the case of the semi-free action, in [21], page 188; assertion 2 follows from the work of Freedman, see [10]; assertion 3 is proved using the retraction Homeo (S^1) into $SO(2)$ given by the Poincaré number, see [7].

Appendix

Lemma. *The extensions:*

$$\begin{array}{ccccccccc} 0 & \to & C_2 & \to & \tilde{A}_5 & \to & A_5 & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & C_2 & \to & A_5 \times A_5 & \to & A_5 \times A_5 & \to & 0 \\ & & & & \downarrow & & \downarrow {\scriptstyle (h, h')} & & \\ 0 & \to & C_2 & \to & SO(4) & \to & SO(3) \times SO(3) & \to & 0 \end{array}$$

are not split, h and h' can be any nontrivial representations of A_5 and f is either $(Id \times \{I\})$ or $(\{I\} \times Id)$.

Proof. Standard theory of group extensions and cohomology (see [4]) allows us to reduce to the:

Proposition. *Any non trivial homomorphism $A_5 \xrightarrow{i} SO(3)$ induces an isomorphism $Z/2 = H^2(BSO(3); Z/2) \xrightarrow{i} H^2(BA_5; Z/2) = Z/2$.*

Proof of the Proposition. If the corresponding extension is split, then $Z/2 \times A_5 \subset S^3$, but $A_5 = 60$ so there exists a $Z/2 \subset A_5$ so $Z/2 \times Z/2$ would act freely on S^3, which cannot happen.