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EXTERIOR ALGEBRAS AND THE QUADRATIC RECIPROCITY LAW

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ABSTRACT. As is known, the theory of exterior algebras can be used to derive the properties of determinants and of the signature of permutations (cf. Chevalley, "Fundamental Concepts of Algebra", N.Y., 1956). We show that the properties of the Jacobi symbol, including reciprocity, can also be derived easily from this source.

In this note we consider the connection between a certain identity for exterior algebras and the quadratic reciprocity law.

It is easily shown that if M is a module then, in the exterior algebra of M,

$$(1) \qquad \bigwedge_{i=1}^{m} \bigwedge_{j=1}^{n} a_{i,j} = (-1)^{\binom{m}{2}\binom{n}{2}} \bigwedge_{j=1}^{n} \bigwedge_{i=1}^{m} a_{i,j}$$

for all $a_{i,j} \in M$. However it is also easily shown that this identity is equivalent, in case m and n are odd and relatively prime, to the reciprocity law for the Jacobi symbol

$$(2) \qquad (n|m) = (-1)^{\frac{m-1}{2}\frac{n-1}{2}} (m|n) \,.$$

This provides a very simple and transparent treatment of quadratic reciprocity. Apart from the formulation in terms of exterior algebras, which though convenient is not essential, this approach is substantially that of Zolotarev (cf. [9], [1], [7]). It is curious that it is so little known considering the attention given in the literature to Zolotarev's Theorem (which appears in [9] as a preliminary to the proof of the reciprocity law).

After definitions and preliminaries in 1, we prove (1), and the equivalence of (1) and (2), in 2. Other formulas such as the two supplementary laws and the second multiplicativity formula are considered briefly in 3, together with