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in  $X'(S)$ . Suppose  $X$  is an Abelian scheme over  $S$  and  $R$  is the subgroup of  $X(S)$  consisting of constant sections of  $X/S$ . Let  $s \in X(S)$ . Then the set  $s + R$  is a set of bounded height.

LEMMA 3.1.1 (Manin). *Suppose  $E$  is a finite dimensional  $K$  vector subspace of  $K(C)$ . Then the set*

$$T = \{s \in C(S) : \exists k \neq 0 \in E \text{ such that } s*k = 0\}$$

*has bounded height.*

*Proof.* Without loss of generality we may increase  $E$  to suppose that the rational map  $g: C \rightarrow \mathbf{P}_K(E)$  given on points by  $x \rightarrow (e \in E \rightarrow e(x))$  is birational onto its image (note:  $g$  is actually a morphism on the complement of the polar locus of  $E$ ). It follows that  $g$  induces an embedding of the generic fiber of  $C/S$  into  $\mathbf{P}_{K(S)}(E \otimes K(S))$ . Let  $h$  denote the logarithmic height with respect to this embedding. It follows that if  $s \in C(S)$ ,  $g \circ s$  is constant or  $g \circ s$  has degree one. In the former case  $h(s)$  is zero and the degree of the Zariski closure of  $g \circ s(S)$  in  $\mathbf{P}(E)$  in the latter.

Now if  $s \in T$ , and  $g \circ s$  is not constant, it follows that the Zariski closure of  $g \circ s(S)$  is a component of a hyperplane section of the Zariski closure of  $g(C)$ . Hence,  $h(s)$  is less than or equal to the degree of the Zariski closure of  $g(C)$ . This proves the lemma.  $\square$

The key property about heights we will need is:

THEOREM 3.1.2. *Suppose  $C \rightarrow S$  is as in the above theorem. If  $C(S)$  contains an infinite set of bounded height then  $C$  is a constant family.*

(See Corollary 2.2, Chapter 8 of [L-FD].)

Hence all we need prove is that the elements of  $C(S)$  have bounded height.

## 2. LANG-SIEGEL TOWERS

Suppose the genus of  $C$  is at least 1. Suppose  $T$  is an infinite subset of  $C(S)$ .

PROPOSITION 3.2.1. *There exists a projective system of curves*

$(\{C_n\}, \{h_{m,n}\}), m, n \in \mathbf{Z}_{>0}$  and  $n \leq m$ , over  $K$  such that

- (i)  $C_1 = C$ ,
- (ii)  $h_{m,n}: C_m \rightarrow C_n$  is étale,

- (iii)  $(h_{m,1})^{-1}(T) \cap C_m(S)$  is infinite,
- (iv) There exists a finite covering  $S_{m,n}$  of  $S$  such that the fiber product of  $h_{m,n}$  with  $S_{m,n}$  is Galois, Abelian and of positive degree.

Let  $J$  denote the Jacobian scheme of  $C$  over  $S$ . Let  $a: C \rightarrow J$  be an Albanese morphism. Let  $p$  be a prime. Let  $\bar{T}$  denote the closure of  $a(T)$  in  $J(S) \otimes \mathbf{Z}_p$ . Since  $a(T)$  is infinite it follows from the Mordell-Weil Theorem that there exists a  $t \in \bar{T} - a(T)$ . Let  $t_n \in T$  such that  $t - a(t_n) \in p^n J(S)$ . Let  $C_n$  denote the normalization of the fiber-product of  $C$  and  $J$  via the map  $H_n: x \rightarrow p^n x + t_n$  and  $h_{n,1}$  the natural map from  $C_n$  to  $C$ . It follows that  $C_n$  is defined over  $S$  and since  $H_m(J(S)) \supseteq \{t_n: m \mid n\}$  that  $h_{n,1}(C_m(S))$  contains an infinite subset of  $T$ .

All that remains is to exhibit the maps  $h_{m,n}$ . Clearly,  $t_m - t_n = p^n r_{m,n}$  for some  $r_{m,n} \in J(S)$ . Let  $H_{m,n}$  denote the map  $x: p^{m-n}x + r_{m,n}$ . Then  $H_{m,k} = H_{n,k} \circ H_{m,n}$ . It follows that  $H_{m,n}$  pulls back to a morphism  $h_{m,n}: C_m \rightarrow C_n$ . It is easy to see that this morphism becomes Abelian after adjoining the  $p^{m-n}$ -torsion points on  $J$ . This proves the proposition.  $\square$

*Remark.* One can also prove the above proposition with the condition  $n \leq m$  replaced by  $n \mid m$ .

### 3. COROLLARIES OF THE THEOREM OF THE KERNEL

LEMMA 3.3.1. *Suppose  $g: X' \rightarrow X$  is a morphism of smooth proper schemes with geometrically connected fibers over  $S$ . Then if  $\mu \in PF(X'/S)$  and  $s, t \in X(S)$ ,  $(g^*\mu)(s, t) = \mu(g \circ s, g \circ t)$ .*

*Proof.* This follows easily from Lemma 1.3.2.  $\square$

Suppose  $J$  is the Jacobian of  $C$  over  $S$  and  $g$  is an Albanese morphism, then since  $g^*: H_{DR}^1(J/S) \rightarrow H_{DR}^1(C/S)$  is an isomorphism  $g^*: PF(J/S) \rightarrow PF(C/S)$  is an isomorphism.

LEMMA 3.3.2. *Let  $\mu$  be a fixed Picard-Fuchs differential equation on  $C/S$ . Then  $\{\mu(s, t): s, t \in C(S)\}$  lies in a finite dimensional subspace of  $K[S]$  over  $K$ .*

*Proof.* Suppose  $\tilde{\mu} \in PF(J/S)$  such that  $g^*\tilde{\mu} = \mu$ . The lemma follows from the Mordell-Weil theorem which together with the Theorem of the kernel implies that  $J(S)$  modulo the kernel of the homomorphism  $s \rightarrow \tilde{\mu}(e, s)$  is a finitely generated Abelian group.  $\square$