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THE POMPEIU PROBLEM REVISITED

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Abstract. One of the central results connected with the Pompeiu problem is a theorem of Brown, Schreiber and Taylor. Using some old work of the authors on spectral synthesis, a proof of this result is given. Though separately dealt with, it is shown that some of the main results for the Pompeiu problem for non-Euclidean symmetric spaces can also be treated in the same spirit. In all the cases the role of representations of the underlying group of isometries is highlighted. This point of view leads to some new results for the Pompeiu problem for two sided translations on the non-commutative groups $SL(2, \mathbf{R})$ and $M(2)$. Finally, a brief discussion is provided for some related problems.

1. Introduction

Let X be a locally compact Hausdorff space and G a group of homeomorphisms of X each of which leaves a given non-negative Radon measure μ invariant. The central theme of this article is what is known in the literature as the Pompeiu property: a relatively compact measurable subset $E \subseteq X$ is said to have the Pompeiu property if for a continuous function f on X,

$$(1.1) \qquad \int_{gE} f(x) d\mu(x) = 0 \quad \text{for all} \quad g \in G$$

implies $f \equiv 0$.

The Pompeiu property in a wide variety of settings and its relation to other problems have been the subject-matter of a large number of investigations beginning with two articles by the Roumanian mathematician D. Pompeiu in 1929 ([18], [19]). In the first paper ([18]) the set-up was essentially $X = \mathbf{R}^2$ with the Lebesgue measure μ and G the group $\mathbf{R}^2$

acting through translations, where the unit disc $D \subseteq \mathbf{R}^2$ was claimed to have the Pompeiu property. We need only to look at the Fourier-Laplace transform of the characteristic function 1_D of the disc D to see that if a is a zero of the Bessel function J_1, then $f(x, y) = \sin ax$ is a nonzero function satisfying (1.1). In fact, it was later realised that no bounded subset E has the Pompeiu property, for this set-up (see [12], Theorem 4.3). However, as already seen in the second paper of Pompeiu ([19]), the problem becomes more meaningful, even hard, if either G is replaced by a larger group or further restrictions are imposed on f in condition (1.1).

The basic paper in the theory is, however, the 1973 work of Brown, Schreiber and Taylor ([12]), who considered $X = \mathbf{R}^2$ with Lebesgue measure and $G = M(2)$, the Euclidean motion group (which, they point out, is no different from the more general setting of $X = \mathbf{R}^n$ and $G = M(n)$). They show that the Pompeiu property is closely related to the work of L. Schwartz on mean periodic functions ([23]) — a key observation for their and subsequent work on this theme. Their main result states that E has the Pompeiu property if and only if the Fourier-Laplace transform of the characteristic function 1_E,

$$(1.2) \qquad \hat{1}_E(z_1, z_2) = \int_E e^{-i(t_1 z_1 + t_2 z_2)} dt_1 dt_2, \ (z_1, z_2) \in \mathbf{C}^2$$

does not vanish identically on any of the varieties

$$M_\alpha = \{(z_1, z_2) \in \mathbf{C}^2 : z_1^2 + z_2^2 = \alpha^2\}, \qquad 0 \neq \alpha \in \mathbf{C}.$$

By direct computation, these authors are able to verify condition (1.2) for all proper ellipses: thus $E = \{(x, y): x^2/a^2 + y^2/b^2 \leqslant 1\}$ with $ab \neq 0$, $a \neq b$ has the Pompeiu property. Such direct computation can work only for sets with rigid geometric properties. Brown, Schreiber and Taylor have a general result: a bounded simply connected domain whose boundary has a "corner" has the Pompeiu property ([12], Theorem 5.11; see Section 3 for a precise statement). Triangles, parallelograms and polygonal figures thus have the Pompeiu property.

Another formulation of the problem (see [3], [33], [34]) is that, if E is a *simply connected bounded domain with a Lipschitz boundary*, then condition (1.2) holds if and only if there is a complex number $\alpha \neq 0$ such that the over-determined boundary value problem

$$\Delta T + \alpha T = 0 \quad \text{on} \quad E$$

$$(1.3) \qquad T = \text{constant} \neq 0 \text{ on } \partial E, \partial T/\partial n \equiv 0 \quad \text{on} \quad \partial E$$

has a solution. S. A. Williams ([34]) used (1.3) to show that if E fails to have the Pompeiu property then ∂E is real-analytic. In yet another development C. A. Berenstein showed that if E is a simply-connected bounded domain with smooth boundary and $\hat{1}_E$ vanishes on an infinite sequence of varieties $M_{\alpha_1}, M_{\alpha_2}, \ldots$, then indeed E is a disc (see [3]).

In the more general setting of a Riemannian symmetric space with the associated Riemannian volume measure and G a group of isometries, the problem has been studied in depth by Berenstein and Zalcman ([9]) and Berenstein and Shahshahani ([7]). Here again, the question reduces to one of spectral analysis in Euclidean spaces. Since Schwartz's theorem holds in dimension one only, definitive results could be obtained for rank-1 symmetric spaces alone. The differential equation approach has been fruitful in this case also.

In this paper we present a brief survey of the development outlined above. Apart from the original paper of Brown, Schreiber and Taylor ([12]), a proof of their main theorem can also be found in Berenstein and Zalcman [9] as a particular case of their more general set-up. In this paper we give a proof of the main theorem of Brown, Schreiber and Taylor and its analogue for non-compact symmetric spaces in line with our approach to the problem of spectral analysis in [1]. This, we believe, has the merit of being more transparent, at least for the somewhat less general form of the theorem that we consider here. Our proof also provides an application of the main results in [1]. We are also able to treat some analogous results of Berenstein and Zalcman ([9]) for symmetric spaces of the compact type in the same spirit. We then consider the Pompeiu problem in the context of the group $SL(2, \mathbf{R})$ and $M(2)$ and derive some results based on their representation theory.

The paper is organised as follows. Instead of trying to unify the treatment of the problem for the symmetric spaces of the three types (compact, non-compact and Euclidean) we choose to present them separately: the spaces of Euclidean type in Section 3, those of non-compact type in Section 5 and the compact case in Section 6. We take care, however, to stress the basic similarity and to bring out the role of the so-called class-1 representations in each case. In Section 2, we discuss spectral analysis of radial functions on $\mathbf{R}^n$ — for use in Section 3. In Section 4, we discuss a conjecture which remains a major open question in the theory. In Section 7, we consider what can be called the Pompeiu problem on groups — an area that remains largely unexplored yet. Finally, in Section 8, we try to provide a brief survey of the literature on some allied problems. Though extensive, our bibliography is far from complete. We refer the reader to the bibliographies in [9], [12] and [35].

2. Spectral analysis of radial functions

We denote by $\mathscr{E}(\mathbf{R}^n)$, the space of C^∞ functions on $\mathbf{R}^n$ with the usual topology and by $\mathscr{E}'(\mathbf{R}^n)$, the dual space of distributions of compact support with the strong topology — both Fréchet-Montel and hence reflexive spaces. $C_c^\infty(\mathbf{R}^n)$ is the space of C^∞-functions of compact support. For a space of functions or distributions $\mathscr{F}$, we denote the usual action of an element σ of the orthogonal group $O(n, \mathbf{R})$ by the notation $f \to f^\sigma$. $\mathscr{F}_{\text{rad}}$ will stand for the space of those $f \in \mathscr{F}$ which are invariant under $O(n, \mathbf{R})$, i.e., $f^\sigma = f$ for all $\sigma \in O(n, \mathbf{R})$. $\mathscr{E}'(\mathbf{R}^n)_{\text{rad}}$ is a closed subspace of $\mathscr{E}'(\mathbf{R}^n)$ and the spaces $\mathscr{E}(\mathbf{R}^n)_{\text{rad}}$ and $\mathscr{E}'(\mathbf{R}^n)_{\text{rad}}$ are (strong) duals of each other. In the case $n = 1$, even functions are the analogues of radial functions and we write $\mathscr{F}_e$ to mean $\mathscr{F}_{\text{rad}}$. Though our considerations in this section hold for all $n \geqslant 2$, we shall restrict ourselves to the case $n = 2$ to keep the exposition simple.

We start with a slightly weaker version of the classical theorem of L. Schwartz ([23]).

THEOREM 2.1 (L. Schwartz's theorem on spectral analysis). *Let $\mathscr{U}$ be a nontrivial closed subspace of $\mathscr{E}(\mathbf{R})$, which is closed under translations, then $\mathscr{U}$ contains an exponential function $e^{i\lambda x}$ for some $\lambda \in \mathbf{C}$.*

As pointed out in [1] an immediate corollary of the theorem is: If $\mathscr{U}$ is a nontrivial closed subspace of $\mathscr{E}(\mathbf{R})_e$ which is closed under convolution against all $T \in \mathscr{E}'(\mathbf{R})_e$, then $\mathscr{U}$ contains a function of the form $\psi_\lambda(x) = (e^{i\lambda x} + e^{-i\lambda x})/2$.

We now introduce a family of functions on $\mathbf{R}^2$ which is central to spectral analysis of radial functions. For $\lambda \in \mathbf{C}$, define

$$\phi_\lambda(x) = \int_{|w|=1} e^{-i\lambda(x\,.\,w)} dw, \; x \in \mathbf{R}^2$$

where the integral is with respect to the normalised Lebesgue measure on the unit circle. Here $x\,.\,w$ is the usual inner product. It is immediate that ϕ_λ is a radial function for each $\lambda \in \mathbf{C}$. For $f \in C_c^\infty(\mathbf{R}^2)_{\text{rad}}$, we define a transform (sometimes called the Bessel transform):