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then the argument is as in Theorem 7.1 and we can prove $\Pi_\lambda(1_E) = 0$. If Π_λ is not irreducible, then depending on n, m we can find a discrete series representation or an irreducible finite dimensional representation Π occurring either as a subrepresentation or as a subquotient of Π_λ for which $\Pi(1_E) = 0$. To do this, we need the exact G-module structure of Π_λ which in the case of $PSL(2, \mathbf{R})$ is available (see for example [14]).

§ 7.3. Consider the case $G = \mathbf{R}^2$ and G acting on itself *by translations*. In this case, Brown, Schreiber and Taylor have proved that there are no Pompeiu sets ([12]). In view of this it would be natural to ask if there are sets E satisfying the conditions of Theorem 7.1 at all. Identify $\mathbf{R}^2$ with G/K where $G = M(2)$ and $K = SO(2, \mathbf{R})$; if $E \subseteq G/K$ then one can show that the condition of Brown, Schreiber and Taylor considered in Section 3 is equivalent to the condition $\Pi_\lambda(1_{\tilde{E}}) \neq 0$ for $\lambda \in \mathbf{C}, \lambda \neq 0$. (A special case of this observation is also made in [30]). Hence by the discussion in Section 3, there are plenty of sets E with this property. As we have seen, topologically $G \approx \mathbf{R}^2 \times SO(2, \mathbf{R})$. We now observe that if E is chosen as above in $\mathbf{R}^2$ and I is a suitably chosen arc in $SO(2, \mathbf{R})$, then the set $F = E \times I$ considered as a subset of G satisfies $\Pi_\lambda(1_F) \neq 0$ for all $\lambda \in \mathbf{C} - \{0\}$ as well as $\chi_n(1_F) \neq 0$ for all $n \in \mathbf{Z}$. (We need only to choose I whose length is irrational modulo 2π.)

8. Concluding remarks

In this paper, we have restricted our attention to the Pompeiu property for a single set E. One can also consider the Pompeiu property for a collection of sets or distributions of compact support as in [9], [12]. There are also closely related properties such as the Morera property — see [12] for details.

As pointed out earlier the Pompeiu problem becomes easier if one considers only integrable functions. Investigations under this assumption have been done, for example, in [2], [20], [24] and [28]. If one only considers integrable functions one need not restrict oneself to relatively compact sets. Moreover, considering integrable functions is equivalent to considering finite complex measures. Thus for G a locally compact abelian group a Borel subset $E \subseteq G$ is said to be a *determining set for finite complex measures* if for a finite complex measure μ on G, $\mu(gE) = 0$ for all $g \in G$ implies $\mu = 0$.

For locally compact abelian groups it is easy to see that a set of finite Haar measure is a determining set for finite complex measures if and only if the Fourier transform $\hat{1}_E$ does not vanish on any nonempty open subset of the dual group $\hat{G}$. Thus bounded Borel subsets of $\mathbf{R}^n$ of positive Lebesgue measure are determining sets by the analyticity of $\hat{1}_E$. Classical quasi-analyticity results apply to give conditions on the growth of an unbounded subset $E \subseteq \mathbf{R}^n$ to be a determining set. Settling a problem that was open for some time, Kargaev ([17]) proved the existence of sets $E \subseteq \mathbf{R}^n$ of finite Lebesgue measure which are not determining sets for finite complex measures.

The problem of determining sets has also been studied with the class of probability measures replaced by other classes of measures, e.g., a class of infinite measures with growth/decay conditions (see [22], [11] and [28]). Also different groups of homeomorphisms acting on X have been considered in these studies.

Finally, we refer to the following form of the *support problem* analogous to the well known problem in the case of Radon transform. Let X be a symmetric space (Euclidean, compact or non-compact). Let x_0 be a fixed point of X. If E is a relatively compact subset of positive measure and if $\int_{gE} f = 0$ for all $g \in G$ with $d(x_0, gx_0) > R$ what can one say about the support of f with respect to the reference point x_0? (Here, d stands for the geodesic distance.) Some partial answers to this question are known (see [26] and [28]).

We have not addressed ourselves in this paper to the situation when X is an infinite-dimensional Hilbert space or X is an arbitrary Riemannian manifold. Another important problem we have not considered is the *local version* of the Pompeiu problem. (For this, we refer the reader to [5] and [6]). We have restricted ourselves to the situation of symmetric spaces and locally compact groups and the relationship of the Pompeiu problem with harmonic analysis and representation theory.

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