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characteristic of the first family, starting at a small distance ε from the point 1 (where the distances of points 1-4 to the origin are of order 1). The endpoint of this characteristic, taken at the level of point 4, lies at some distance from the singular characteristic 14, namely, at a distance $a\varepsilon + b\varepsilon^2 ln\varepsilon + \dots$. The logarithmic term describes the scattering at the origin: in a regular family the distance would be $a\varepsilon + b\varepsilon^2 + \dots$.

In the 3-dimensional physical space (i.e. for space-time of dimension 4) generic wave fronts (travelling in inhomogeneous media and governed by a variational principle) acquire singular lines, connecting them with waves of different kinds and moving with the wave fronts.

It is interesting to note that the case $n = 1$ is more difficult than $n > 1$. The results are at present formal in both cases. They probably hold for the C^∞ problem both for $n = 1$ and $n > 1$. The divergence of the normalizing series in the analytical problem is proven in the case $n = 1$, while for $n > 1$ there exists still some hope that the series converges. The qualitative results, described above, are independent of the convergence of the series: we need only finite segments of the series.

§4. Legendre fibrations and singularities

The simplest examples of Legendre fibrations are the projectivized cotangent bundles

$$PT^*V^n \to V^n$$

and the "forgetting of derivatives" mappings

$$J^1(M, \mathbf{R}) \to J^0(M, \mathbf{R})$$

(in coordinates: $(p, q, y) \mapsto (q, y)$).

Definition. A *Legendre fibration* is a fibration of a contact manifold with Legendre fibres.

THEOREM. *All the Legendre fibrations of a given dimension are locally contactomorphic (locally = in a neighbourhood of any point of the total space).*

To prove this theorem it is sufficient to construct a local isomorphism of an arbitrary Legendre fibration with one of the preceding examples.

Let us project a contact hyperplane from its contact point in the total space of the fibration to a base point. The image is a contact element (a hyperplane in the tangent space to the base of the fibration) since the tangent plane to the Lagrangian fibre lies in the contact hyperplane.

Thus we have defined a mapping from the space of an arbitrary contact fibration to the space of the contact elements of its base (that is, to the space of the projectivized cotangent bundle of the base space).

This mapping transforms fibres into fibres (over the same points). The nondegeneracy of the initial contact structure implies that this mappping is nondegenerate (it is a local diffeomorphism). And it is easy to see that the initial contact structure and the natural contact structure of the contact elements' space agree.

Thus we have obtained a unique local normal form of any Legendre fibration. At the same time we have defined a *natural projective structure* in the fibres of any Legendre fibration.

This projective structure is a contact analogue of the natural affine structure of the fibres of Lagrange fibration in symplectic geometry; this affine structure is the main ingredient of the proof of the Liouville theorem on the invariant tori of integrable Hamiltonian systems.

The projective structure of the Legendre fibres is even better than the affine structure of the Lagrange fibres. Indeed, a diffeomorphism of the base of a Legendre fibration induces a well defined mapping of the fibres (since it acts on the contact elements of the base).

A diffeomorphism of the base of a Lagrangian fibration can be (locally) lifted to a fibred symplectomorphism of the total space, but this lifting is not unique (this ambiguity implies some global annoyances).

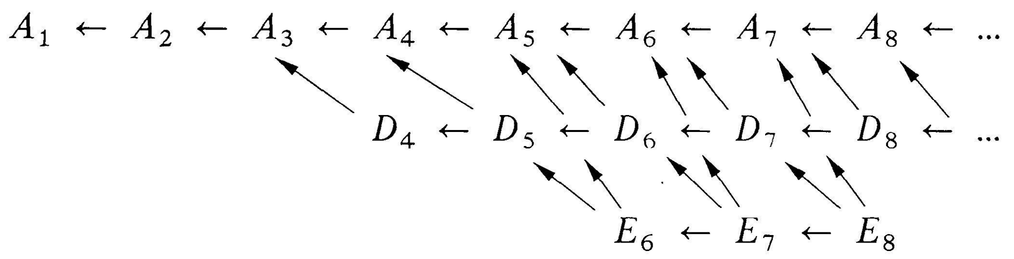
According to the above theorem the Legendre singularities (the germs of the triples $L \hookrightarrow E \twoheadrightarrow B$ consisting of a Legendre embedding and of a Legendre fibration) can be modelled by the Legendre submanifolds of a projectivized cotangent bundle of any manifold, say — of the projective space. The Legendre singularity is defined by its front, if it is a hypersurface (and they usually are).

Now it is easy to deduce that all the Legendre singularities are (locally) equivalent to singularities of Legendre transformations of smooth functions, or of the dual hypersurfaces of smooth projective hypersurfaces or the equidistants of smooth hypersurfaces (and so on).

A Legendre singularity is called *simple,* if all the neighbouring Legendre singularities belong to a finite set of Legendre equivalence classes.

A Legendre mapping is Legendre stable if all the neighbouring Legendre mappings are equivalent to the given one. A similar definition for germs allows a small shift of the origin of the germ: any neighbouring Legendre *mapping* has a Legendre equivalent *germ* at a neighbouring point.

THEOREM. *All the simple and stable analytic Legendre singularities are classified by the simple Lie algebras of types* A, D *and* E:



Namely, the corresponding fronts are **C**-*diffeomorphic to the corresponding discriminants (the sets of nonregular orbits of the corresponding Weyl groups).*

Example. The Weyl group A_μ is the group generated by the reflections of the space $\mathbf{C}^\mu = \{z \in \mathbf{C}^{\mu+1} : z_0 + \ldots + z_\mu = 0\}$ in the diagonal mirrors $z_i = z_j$. The orbits are the unordered $\mu + 1$-tuples $\{z_0, \ldots, z_\mu\}$, such that $z_0 + \ldots + z_\mu = 0$.

The space of orbits is the space of polynomials

$$z^{\mu+1} + \lambda_1 z^{\mu-1} + \ldots + \lambda_\mu .$$

The irregular orbits correspond to polynomials having multiple roots.

For instance, the set of irregular orbits for A_2 is a semicubical parabola in the plane formed by the polynomials

$$\{z^3 + \lambda_1 z + \lambda_2 = (z+t)^2(z-2t)\} .$$

The discriminant for A_3 is the swallowtail surface

$$\{z^4 + \lambda_1 z^2 + \lambda_2 z + \lambda_3 = (z+t)^2 \ldots\} .$$

THEOREM. *The generic Legendre mappings* $L^n \to E^{2n+1} \to B^{n+1}$ *of Legendre manifolds of dimension* $n < 6$ *are simple and stable at all their points (and hence are described by the preceding theorem).*

The classification of the stable Legendre mappings of any dimension up to Legendre equivalence is equivalent to the classification of families of functions up to "V-equivalence" (= fibred diffeomorphisms of the zero level hypersurfaces).

It is also known that the set of topologically different singularities of generic Legendre mappings remains finite for any finite n (Varchenko, Looijenga). However this topological classification is unknown even for those small values of n between 6 and 11, for which there exists an explicit classification up to smooth Legendre equivalence (this classification is described in the last chapter of volume 1 of the book [AGV]).

In the theory of propagation of waves one encounters, besides the usual wave fronts, a Legendre singularity of higher dimension — its front is the graph of the "multivalued time function", whose level sets are the momentary fronts.

Let us consider the positions of a moving front in different moments of time as a hypersurface in space-time. This hypersurface is called the *big front*. The big front is a front of a Legendre mapping over space-time. The *momentary fronts* are its sections by the isochrones (isochrones are the level sets of the time function in space-time).

To study the perestroikas[1]) of the momentary wave fronts we need to reduce the time function to a normal form in a neighbourhood of a singular point of the big front by a diffeomorphism preserving the big front.

In the case when the big singularity is simple and stable, this can be done very explicitly, using the technique of the invariant theory of Weyl groups (or of Coxeter groups).

The main ingredient is the study of vector fields, tangent to the discriminant. Such vector fields form a module over the algebra of functions. Hence the knowledge of few particular fields, tangent to the discriminants, permits one to construct many diffeomorphisms preserving the discriminant. Using these diffeomorphisms one can reduce the time function in a neighbourhood of the origin of the space of orbits of a Weyl group to a linear normal form. The corresponding linear function on the space of orbits is an invariant of degree 2 (considered as a function on the space of orbits).

For instance, a generic function in a neighbourhood of the most singular point $\lambda = 0$ of the generalized swallowtail $\{\lambda : \exists t : x^{\mu+1} + \lambda_1 x^{\mu-1} + \ldots + \lambda_\mu = (x+t)^2 \ldots \forall x\}$ is reducible to the normal form $\pm\lambda_1$ by a swallowtail preserving diffeomorphism.

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[1]) In Russian the word perestroika was always used in this mathematical sense, for instance "Morse surgery" is "Morse perestroika" in Russian. In past translations from the Russian, "perestroika" of wave fronts was called "metamorphosis", but now I may use the international word "perestroika".

One may find in the literature the statement that the local perestroikas of the wavefronts generated by the general Legendre mappings over space-time and of the equidistants of the smooth hypersurfaces are the same. It seems this has never been correctly proved. It is now known (Nay, Tchekanov) that the corresponding statement for the caustic perestroikas is wrong. The fact that the moving Lagrange manifold lies in a (moving) hypersurface of the cotangent bundle space, which is quadratically convex along the fibres, implies some topological restrictions on the local perestroikas of the caustics.

The contact geometry analogues of these results have not yet been formulated (one of the variants deals with the Legendre submanifolds of a hypersurface in the projective cotangent fibration space, which is locally quadratically convex in the sense of the projective structure of the fibres).

The local classification of generic Legendre singularities is the base of a global theory of *Legendre cobordisms* and *characteristic classes.*

Let us consider the projectivized (or the spherized) cotangent bundle $E(M)$ of a manifold M with boundary ∂M. A Legendre submanifold L of M, which is transversal to ∂E, has a "Legendre boundary", which is an immersed Legendre submanifold of $E(\partial M)$. It is defined by "section and projection": first we intersect L with the hypersurface ∂E, and then we project the intersection along the characteristics of ∂E to the space of characteristics, which is $E(\partial M)$. The projection has dimension $\dim L - 1$ and is a Legendre submanifold of $E(\partial M)$.

This Legendre boundary construction gives birth to many cobordism theories since we can consider oriented or non-oriented bases and immersed Legendre manifolds, formed by cooriented or noncooriented contact elements or by jets of functions on manifolds with boundary.

Example 1. The group of cobordism classes of (a) oriented, (b) nonoriented Legendre submanifolds in the space of cooriented contact elements of the plane is isomorphic to (a) the group of integers, $\mathbf{Z}$, the generator being an eight-shaped curve with 2 cusps at the top and at the bottom; (b) to the trivial group.

Example 2. The cobordism of the oriented generator to zero shows that this generator is the Legendre boundary of a Legendre Möbius band over a halfplane M. This construction defines a Legendre embedding of a Klein bottle in $\mathbf{R}^5$. (See [A2].)

For more details on the Legendre cobordisms and characteristic classes consult the book [Va] by V.A. Vassilyev.

Example 3 (M. Audin). The classes of nonoriented Legendre cobordisms in the spaces of 1-jets of functions in the spaces $\mathbf{R}^n$ form a skew-commutative graded ring, which is isomorphic to the graded ring $\mathbf{Z}_2[x_5, x_9, x_{11}, \ldots]$ of polynomials with coefficients in $\mathbf{Z}_2$ and with arguments x_k of odd degrees $k \neq 2^r - 1$.

In the oriented case the ring is isomorphic to the exterior algebra over $\mathbf{Z}$ with generators of degrees $1, 5, 9, \ldots, 4n + 1, \ldots$ mod torsion.

The proofs are based on the Eliashberg reduction of the problem to the calculation of the homotogy groups of the Thom spectra of the tautological bundles over the Lagrangian Grassmannians (the details are in the Eliashberg paper [El]).

On the other side the classification of Legendre singularities generates a complex, whose cells are singularity types and whose boundaries are defined by the adjacency of the singularities. The initial parts of these complexes were calculated by V.A. Vassilyev (see his book [Va]). The cohomology of these complexes defines Legendre characteristic classes (the simplest of them is the Maslov class). These classes can be generated also by the corresponding universal spaces (the Lagrangian Grassmannians $U(n)/O(n)$).

But the information on the singularities' coexistence, compressed in the Vassilyev complexes of singularities and of multisingularities is not reduced to the calculation of the characteristic numbers in terms of the singularities.

Example. The number of A_3 points on a generic closed Legendre surface immersed in $J^1(M^2, \mathbf{R})$ is always even. The number of intersections of the strata (A_1A_2), (A_1A_4), (A_2A_4), (A_1A_6), $(A_1A_2A_4)$ of the front are (mod 2) characteristic numbers for the Legendre mappings.

The number of singularities of any given type on the Legendre boundary is even. For the Legendre boundary of an oriented manifold the numbers of singularities E_6, (or E_7 or E_8) counted with some sign convention, are equal to zero.

For Legendre immersions in the space of 1-jets of functions, Vassilyev has defined orientation rules, for which $\# A_5 = 0$ (the number of A_5 singularities on a closed oriented Legendre 4-manifold is equal to zero), $\# A_6 = \# E_6$, $\# E_7 + 3\# E_8 = 0$. The class A_5 defines a cohomology class in the Vassilyev complex, but it is not realizable by a Legendre immersion.

The topology of Legendre immersions and embeddings is far from being settled.