Zeitschrift: L'Enseignement Mathématique
Band: 36 (1990)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: YANGIANS AND R-MATRICES
Autor: Chari, Vyjayanthi / Pressley, Andrew
DOI: https://doi.org/10.5169/seals-57909

L'Enseignement Mathématique, t. 36 (1990), p. 267-302

YANGIANS AND R-MATRICES

by Vyjayanthi CHARI and Andrew PRESSLEY

0. INTRODUCTION

Quantum groups arose as the natural language in which to formulate certain techniques which had been developed to construct and solve integrable quantum systems (see [6]). The most important examples are quantum Kac-Moody algebras and Yangians. Representations of both types of quantum groups are closely related to solutions of the quantum Yang-Baxter equation (QYBE). In particular, finite-dimensional irreducible representations of Yangians give rational solutions of the QYBE. In this paper we shall give a complete and elementary description of all the irreducible finite-dimensional representations of the Yangian associated to $\mathfrak{sl}_2$. The importance of this example is analogous to that of $\mathfrak{sl}_2$ itself, whose representation theory is the foundation for that of an arbitrary semi-simple Lie algebra.

A quantum group is a deformation of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ in the category of (not necessarily co-commutative) Hopf algebras. More precisely, let $\mathbf{C}[[h]]$ be the algebra of power series in an indeterminate h; we shall have occasion to use the grading on $\mathbf{C}[[h]]$ obtained by setting $\deg h = 1$. Then, a quantum group is a Hopf algebra A over $\mathbf{C}[[h]]$ such that one has an isomorphism of Hopf algebras

$$A/hA \cong U(\mathfrak{g}) \ . \tag{0.1}$$

Further, A is required to be complete and topologically free as a $\mathbf{C}[[h]]$-module (the latter condition means that $A/h^n A$ is a free $\mathbf{C}[[h]]/(h^n)$-module for all $n \geqslant 1$). We shall sometimes refer to A as a quantization of $U(\mathfrak{g})$. One thinks of A as a "quantum" object and interprets the isomorphism (0.1) as meaning that $U(\mathfrak{g})$ is obtained from A by taking the "classical limit $h \to 0$".

Let $\Delta: A \to A \otimes A$ be the co-multiplication map of A, $\sigma: A \otimes A \to A \otimes A$ the switch of the two factors, and set $\Delta' = \sigma\Delta$. For any $x \in U(\mathfrak{g})$, choose $a \in A$ such that $a \equiv x (\operatorname{mod} h)$. Then

$$\frac{\Delta(a) - \Delta'(a)}{h} \ (\operatorname{mod} h)$$

is a well-defined element $\delta(x) \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$. The map $\delta: U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is determined by its restriction to $\mathfrak{g}$, which maps $\mathfrak{g}$ into $\mathfrak{g} \otimes \mathfrak{g}$, by the formula

$$\delta(ab) = \delta(a)\Delta(b) + \Delta(a)\delta(b) .$$

Moreover, δ is a 1-cocycle on $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$, and the dual map $\delta^*: \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ gives $\mathfrak{g}^*$ the structure of a Lie algebra. An important special case is that in which δ is a 1-coboundary, which means that, for some $r \in \mathfrak{g} \otimes \mathfrak{g}$ we have

$$\delta(x) = [x \otimes 1 + 1 \otimes x, r]$$

for all $x \in \mathfrak{g}$. The dual of this map δ defines a Lie bracket on $\mathfrak{g}^*$ if and only if

$$r^{12} + r^{21} \in \mathfrak{g} \otimes \mathfrak{g} \tag{0.2}$$

and

$$< r, r > \equiv [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \tag{0.3}$$

are $\mathfrak{g}$-invariant. Here, $r^{12} = r \otimes 1 \in U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g})$ etc. In particular, equations (0.2) and (0.3) are satisfied if r is skew and satisfies the classical Yang-Baxter equation (CYBE)

$$<r, r> = 0 .$$

Yangians arise from the case $\mathfrak{g} = \mathfrak{a}[t]$, where $\mathfrak{a}$ is a finite-dimensional complex simple Lie algebra and t an indeterminate. Note that setting $\deg t = 1$ makes $U(\mathfrak{g})$ a graded Hopf algebra. Then, r becomes a function of two variables t_1, t_2 with coefficients in $\mathfrak{a} \otimes \mathfrak{a}$; the skewness condition is now

$$r^{12}(t_1 - t_2) + r^{21}(t_2 - t_1) = 0 , \tag{0.4}$$

and the CYBE becomes

$$[r^{12}(t_1 - t_2), r^{13}(t_1 - t_3)] + [r^{12}(t_1 - t_2), r^{23}(t_2 - t_3)] + [r^{13}(t_1 - t_3), r^{23}(t_2 - t_3)] = 0 . \tag{0.5}$$

The simplest solution of equations (0.4) and (0.5) is

$$r(t_1, t_2) = \frac{\Omega}{(t_1 - t_2)} , \tag{0.6}$$

where

$$\Omega = \sum_{\lambda} (I_\lambda \otimes I_\lambda) , \tag{0.7}$$

and $\{I_\lambda\}$ is an orthonormal basis of $\mathfrak{a}$ with respect to a fixed $\mathfrak{a}$-invariant bilinear form on $\mathfrak{a}$. According to Drinfel'd [4], there is, up to isomorphism, a unique quantization A of $U(\mathfrak{a}[t])$ such that:

(1) A is a graded $\mathbf{C}[[h]]$-algebra and (0.1) is an isomorphism of graded algebras;

(2) $\delta(x) = [x \otimes 1 + 1 \otimes x, r]$ for $x \in \mathfrak{a}[t]$, where r is given by (0.6).

The algebra A is generated by elements x, $J(x)$ for $x \in \mathfrak{g}$, whose "classical limits" are the generators x, xt of $\mathfrak{a}[t]$ (see Definition 1.1 for a precise description of A). The defining relations of A only involve polynomials in h, and hence it makes sense to specialize to a particular value of h. The resulting Hopf algebra A_h over $\mathbf{C}$ is easily seen to be independent of h, up to isomorphism, as long as $h \neq 0$; setting $h = 1$ gives the Yangian $Y(\mathfrak{a})$. (More precisely, $Y(\mathfrak{a}) = A'/(h-1)A'$, where A' is the algebraic direct sum of the homogeneous components of A.)

For the Lie algebra $\mathfrak{a}[t]$, one has the evaluation homomorphisms $\varepsilon_a : \mathfrak{a}[t] \to \mathfrak{a}$ for any $a \in \mathbf{C}$. Pulling back a representation of $\mathfrak{a}$ by such a map gives a so-called "evaluation representation" of $\mathfrak{a}[t]$, and it is known [1], [2] that every finite-dimensional irreducible representation of $\mathfrak{a}[t]$ is isomorphic to a tensor product of evaluation representations. When $\mathfrak{a} = \mathfrak{sl}_2$, the evaluation homomorphisms admit "quantizations" $\varepsilon_a : Y(\mathfrak{sl}_2) \to U(\mathfrak{sl}_2)$ such that

$$\varepsilon_a(x) = x, \qquad \varepsilon_a(J(x)) = ax .$$

Evaluation representations of $Y(\mathfrak{sl}_2)$ can now be defined just as for $\mathfrak{sl}_2[t]$. One of the main results of this paper is:

THEOREM. *Every finite-dimensional irreducible representation of $Y(\mathfrak{sl}_2)$ is isomorphic to a tensor product of evaluation representations.*

The representation theory of $Y(\mathfrak{a})$ is closely related to the quantum Yang-Baxter equation (QYBE):

$$\begin{aligned} &R^{12}(t_1 - t_2)R^{13}(t_1 - t_3)R^{23}(t_2 - t_3) \\ &\quad = R^{23}(t_2 - t_3)R^{13}(t_1 - t_3)R^{12}(t_1 - t_2) . \end{aligned} \tag{0.8}$$

Here, the function $R(t)$ is usually understood to take values in $\mathrm{End}(V \otimes V)$ for some finite-dimensional vector space V, although it makes sense when R takes values in $B \otimes B$ for any associative algebra B. In view of equation (0.6), it is natural to look for solutions of the form

$$R(t) = 1 + t^{-1}(\rho(I_\lambda) \otimes \rho(I_\lambda)) + \sum_{k=2}^{\infty} R_k t^{-k} \tag{0.9}$$

for some representation ρ of $\mathfrak{sl}_2$ on V and some $R_k \in \mathrm{End}(V \otimes V)$. If $\tilde{\rho}$ is an extension of ρ to $Y(\mathfrak{a})$ and $\mathfrak{R} \in Y(\mathfrak{a}) \otimes Y(\mathfrak{a})$ satisfies equation (0.8), then $R = (\tilde{\rho} \otimes \tilde{\rho})(\mathfrak{R})$ will be a solution of the QYBE with values in $\mathrm{End}(V \otimes V)$. Drinfel'd proved [4] that there exists an essentially unique "universal R-matrix" $\mathfrak{R}$, that the resulting matrix-valued solutions $R(t)$ are rational, and that every rational solution of the QYBE of the form (0.9) arises in this way.

Unfortunately, the universal R-matrix for $Y(\mathfrak{a})$ is not known explicitly, so in section 5 we shall give an alternative construction of rational solutions of the QYBE, which relates them to intertwining operators between tensor products of certain representations of $Y(\mathfrak{a})$. Although this is presumably well-known, it does not seem to have appeared in print before. We use this technique to write down explicitly the solutions of the QYBE associated to all the finite-dimensional irreducible representations of $Y(\mathfrak{sl}_2)$. These solutions were first written down by Kulish, Reshetikhin and Sklyanin [9], but without proof (according to these authors "the proof... is lengthy"). We obtain the R-matrices with minimal computation, and the fact they satisfy the QYBE is a consequence of general results.

The relation between R-matrices and Yangians can be inverted. Let $R(t)$ be a rational solution of the QYBE arising from an irreducible representation of $Y(\mathfrak{a})$ on $\mathbf{C}^n$. To this one associates a Hopf algebra Y_R generated by elements $\{t_{ij}^{(k)}\}$, $1 \leqslant i, j \leqslant n$, $k = 1, 2, \ldots$ Let $T(s)$ be the matrix

$$T(s)_{ij} = \delta_{ij} + \sum_k t_{ij}^{(k)} s^{-k} .$$

Then, the relations are

$$(T(t) \otimes id)\,(id \otimes T(s))R(t-s) = R(t-s)\,(id \otimes T(s))\,(T(t) \otimes id)\,,$$

and the co-multiplication map is given by

$$\Delta(T(s)_{ij}) = \sum_k T(s)_{ik} \otimes T(s)_{kj} .$$

Then, $Y(\mathfrak{a})$ is a quotient of Y_R by an ideal generated by certain group-like elements of the centre of Y_R (called "quantum determinants"). This approach to Yangians appears implicitly in the early work on quantum inverse scattering theory (see [6] for an excellent survey) and also explicitly in more recent work (see, for example, Kirillov and Reshetikhin [7]).

We conclude this introduction with some remarks on the literature. There is a classification of the finite-dimensional irreducible representations of Yangians analogous to that for complex semisimple Lie algebras; this will be described in section 2 and used to obtain our main theorem, which provides a concrete model for the representations. The classification was first obtained by Tarasov [12], [13], for the case of $Y(\mathfrak{sl}_2)$, using ideas of Korepin [8], and was extended by Drinfel'd [5] to the general case. The evaluation representations and their tensor products appeared implicitly in the work of Kulish, Reshetikhin and Sklyanin [9] mentioned above, but they did not prove that all finite-dimensional irreducible representations are of this form. Our determination of the precise conditions under which such tensor products are irreducible is also new.

One of the difficulties of this subject is the unfamiliarity of the language in which many of the fundamental papers are written, which is that of quantum inverse scattering theory and exactly solvable models in statistical mechanics. We have tried in our presentation to express the results in more conventional mathematical language. In fact, all that is required is some familiarity with the basic techniques of Lie theory.

1. Yangians

We begin with the definition of the Yangian taken from [4]. Let $\{I_\lambda\}$ be an orthonormal basis of $\mathfrak{sl}_2$ with respect to some invariant inner product $(\, , \,)$; for example, using the trace form

$$(x, y) = \operatorname{trace}(xy) ,$$

one can take the basis $\left\{\frac{x^+ + x^-}{\sqrt{2}}, \frac{i(x^+ - x^-)}{\sqrt{2}}, \frac{h}{\sqrt{2}}\right\}$, where $\{x^+, x^-, h\}$ is the usual basis:

$$[h, x^\pm] = \pm 2x^\pm , \quad [x^+, x^-] = h .$$

Definition 1.1. The *Yangian* $Y = Y(\mathfrak{sl}_2)$ associated to $\mathfrak{sl}_2$ is the Hopf algebra over $\mathbf{C}$ generated (as an associative algebra) by $\mathfrak{sl}_2$ and elements $J(x)$ for $x \in \mathfrak{sl}_2$ with relations

(1) $[x, J(y)] = J([x, y]), \quad J(ax + by) = aJ(x) + bJ(y), \; a, b \in \mathbf{C}$,

(2) $$[J(x), J([y, z])] + \text{cyclic permutations of } x, y, z$$
$$= ([x, I_\lambda], [[y, I_\mu], [z, I_\nu]]) \{I_\lambda, I_\mu, I_\nu\} ,$$

(3) $$[[J(x), J(y)], [z, J(w)]] + [[J(z), J(w)], [x, J(y)]]$$
$$= (([x, I_\lambda], [[y, I_\mu], [[z, w], I_\nu]]) + ([z, I_\lambda], [[w, I_\mu], [[x, y], I_\nu]])) \{I_\lambda, I_\mu, I_\nu\},$$

where repeated indices are summed over and

$$\{x_1, x_2, x_3\} = \sum_\pi x_{\pi(1)} x_{\pi(2)} x_{\pi(3)} ,$$

the sum being over all permutations π of $\{1, 2, 3\}$. The co-multiplication map of Y is given by

$$\text{(4)} \qquad \Delta(x) = x \otimes 1 + 1 \otimes x\,,$$
$$\Delta(J(x)) = J(x) \otimes 1 + 1 \otimes J(x) + \frac{1}{2}[x \otimes 1, \Omega]\,,$$

where Ω is as defined in equation (0.7).

Remarks.

1. Relation (2) is actually a consequence of the other relations. We have kept it because relations (1), (2) and (3) are the defining relations of the Yangian associated to an arbitrary finite-dimensional Lie algebra $\mathfrak{g}$, and in the general case (2) is not redundant.

2. The relations depend on the choice of inner product (,) but, up to isomorphism, the Hopf algebra Y does not.

There is another realization of Y due to Drinfel'd [5, Theorem 1] which we shall need in the discussion of highest weight representations of Y in the next section.

THEOREM 1.2. *Y is isomorphic to the associative algebra over* $\mathbf{C}$ *with generators* x_k^+, x_k^-, h_k *for* $k = 0, 1, \ldots$ *and relations*

(1) $[h_k, h_l] = 0, \quad [h_0, x_k^\pm] = \pm 2x_k^\pm, \quad [x_k^+, x_l^-] = h_{k+l}\,;$

(2) $[h_{k+1}, x_l^\pm] - [h_k, x_{l+1}^\pm] = \pm (h_k x_l^\pm + x_l^\pm h_k)\,;$

(3) $[x_{k+1}^\pm, x_l^\pm] - [x_k^\pm, x_{l+1}^\pm] = \pm (x_k^\pm x_l^\pm + x_l^\pm x_k^\pm)\,.$

The isomorphism ϕ *between the two realizations of* Y *is given by*

$$\phi(h) = h_0, \quad \phi(x^\pm) = x_0^\pm\,,$$
$$\phi(J(h)) = h_1 + \frac{1}{2}(x_0^+ x_0^- + x_0^- x_0^+ - h_0^2)\,,$$
$$\phi(J(x^\pm)) = x_1^\pm - \frac{1}{4}(x_0^\pm h + h x_0^\pm)\,.$$

One of the difficulties which arises in using this realization of Y is that no explicit formula for the co-multiplication map Δ on the generators $h_k, x_k^\pm$ is known. However, the following formulas follow easily from the formulae in Definition 1.1 and Proposition 1.2:

$$\Delta(h_0) = h_0 \otimes 1 + 1 \otimes h_0\,,$$
$$\Delta(h_1) = h_1 \otimes 1 + h_0 \otimes h_0 + 1 \otimes h_1 - 2x_0^- \otimes x_0^+$$
$$\Delta(x_0^+) = x_0^+ \otimes 1 + 1 \otimes x_0^+\,,$$

(1.3)
$$\Delta(x_1^+) = x_1^+ \otimes 1 + 1 \otimes x_1^+ + h_0 \otimes x_0^+$$
$$\Delta(x_0^-) = x_0^- \otimes 1 + 1 \otimes x_0^- ,$$
$$\Delta(x_1^-) = x_1^- \otimes 1 + 1 \otimes x_1^- + x_0^- \otimes h_0 .$$

As an application of these formulas we shall prove the following useful result.

PROPOSITION 1.4. *The assignment* $x_k^+ \mapsto x_k^-, x_k^- \mapsto x_k^+, h_k \mapsto h_k, k \in \mathbf{Z}_+$, *extends to an anti-homomorphism* $\omega \colon Y \to Y$. *Moreover, the following diagram is commutative:*

$$\begin{array}{ccc} Y & \overset{\Delta'}{\to} & Y \otimes Y \\ \omega \downarrow & & \downarrow \omega \otimes \omega \\ Y & \underset{\Delta}{\to} & Y \otimes Y \end{array}$$

Proof. The fact that ω extends to an anti-homomorphism of Y follows almost immediately from the relations in Theorem 1.2. To prove that the diagram is commutative, it is enough to check that $\Delta\omega$ and $(\omega \otimes \omega)\Delta'$ agree on a set of generators of Y. From the relations in (1.2) and the form of the isomorphism ϕ, it is clear that Y is generated by $h_0, x_0^\pm$ and $x_1^\pm$. For $h_0, x_0^\pm$ the verification is trivial. From equations (1.3) we have

$$\begin{aligned} \Delta\omega(x_1^-) &= \Delta(x_1^+) \\ &= x_1^+ \otimes 1 + 1 \otimes x_1^+ + h_0 \otimes x_0^+ . \end{aligned}$$

On the other hand,

$$\begin{aligned} (\omega \otimes \omega)\Delta'(x_1^-) &= (\omega \otimes \omega)\,(x_1^- \otimes 1 + 1 \otimes x_1^- + h_0 \otimes x_0^-) \\ &= x_1^+ \otimes 1 + 1 \otimes x_1^+ + h_0 \otimes x_0^+ . \end{aligned}$$

The proof for x_1^+ is similar.

Definition 1.5. Let H (resp. $N^\pm$) denote the subalgebra of Y generated by the h_k (resp. $x_k^\pm$) for $k \in \mathbf{Z}_+$.

We shall now give a more precise description of the co-multiplication map.

PROPOSITION 1.6. *The co-multiplication map* Δ *of* Y *satisfies:*

(1)
$$\Delta(h_k) = h_k \otimes 1 + h_{k-1} \otimes h_1 + h_{k-2} \otimes h_1 + \cdots + h_0 \otimes h_{k-1} + 1 \otimes h_k \quad \textit{modulo} \ \textstyle\sum_{p \geq 0} Y \otimes Yx_p^+ + Yx_p^+ \otimes Y ;$$

(2) $$\Delta(x_k^+) = x_k^+ \otimes 1 + h_0 \otimes x_{k-1}^+ + h_1 \otimes x_{k-2}^+ + \cdots + h_{k-1} \otimes x_0^+ + 1 \otimes x_k^+ \ \textit{modulo} \ \sum_{p,q,r\geqslant 0} Yx_p^- \otimes Yx_q^+ x_r^+ \ ;$$

(3) $$\Delta(x_k^-) = x_k^- \otimes 1 + x_{k-1}^- \otimes h_0 + x_{k-2}^- \otimes h_1 + \cdots + x_0^- \otimes h_{k-1} + 1 \otimes x_k^- \ \textit{modulo} \ \sum_{p,q,r\geqslant 0} Yx_p^- x_q^- \otimes x_r^+ \ .$$

For the proof, we shall need

LEMMA 1.7. *For all* $k, l \in \mathbf{Z}_+$, *we have* $x_k^+ h_l \in HN^+$ *and* $h_l x_k^- \in N^- H$.

Proof. We prove the first formula by induction on l; the second follows from the first by Proposition 1.4. If $l = 0$, then by (1.2) (1),

$$x_k^+ h_0 = h_0 x_k^+ - 2x_k^+$$

which is in HN^+. Next, by (1.2) (2),

$$\begin{aligned}[h_{l+1}, x_k^+] &= [h_l, x_{k+1}^+] + h_l x_k^+ + x_k^+ h_l \\ &= h_l(x_{k+1}^+ + x_k^+) + (x_k^+ - x_{k+1}^+)h_l \, .\end{aligned}$$

Hence,

$$x_k^+ h_{l+1} = h_{l+1}x_k^+ - h_l(x_{k+1}^+ + x_k^+) + (x_{k+1}^+ - x_k^+)h_l \, ,$$

which belongs to HN^+ by the induction hypothesis.

Proof of Proposition 1.6. It is enough to prove formula (2). For (3) follows from (2) by Proposition 1.4. Also,

$$\begin{aligned}\Delta(h_k) &= \Delta([x_k^+, x_0^-]) \\ &= [\Delta(x_k^+), x_0^- \otimes 1 + 1 \otimes x_0^-] \\ &= h_k \otimes 1 + h_{k-1} \otimes h_0 + \cdots + 1 \otimes h_k - 2\sum_{i=0}^{k-1} x_i^- \otimes x_{k-i+1}^+ \\ &\qquad \textit{modulo} \ \sum_{p,q,r\geqslant 0} [Yx_p^- \otimes Yx_p^+ x_r^+, x^- \otimes 1 + 1 \otimes x^-] \, .\end{aligned}$$

To prove (1), it therefore suffices to prove that $x_q^+ x_r^+ x_0^- \in \sum_{s\geqslant 0} Yx_s^+$. Since

$$x_q^+ x_r^+ x_0^- = x_q^+ h_r + x_q^+ x_0^- x_r^+ \, ,$$

this follows from Lemma (1.7).

To prove (2), define $\tilde{h}_1 = h_1 - \frac{1}{2}h_0^2$. Then:

(1.8) $$\Delta(\tilde{h}_1) = \tilde{h}_1 \otimes 1 + 1 \otimes \tilde{h}_1 - 2x_0^- \otimes x_0^+ ,$$

(1.9) $$[\tilde{h}_1, x_k^+] = 2x_{k+1}^+ ,$$

(1.10) $$[\tilde{h}_1, x_k^-] = -2x_{k+1}^- .$$

In fact, (1.8) follows from (1.3) and (1.9) is proved by induction on k, using the relation

$$[h_1, x_k^+] - [h_0, x_{k+1}^+] = h_0 x_k^+ + x_k^+ h_0 ,$$

the right-hand side of which is $\left[\frac{1}{2} h_0^2, x_k^+\right]$. Finally, (1.10) follows from (1.9) and (1.4).

The proof of (2) now proceeds by induction on k. The result is known for $k = 0$ and 1. For the inductive step, we use (1.8) to obtain

$$\begin{aligned} 2\Delta(x_{k+1}^+) &= \Delta([\tilde{h}_1, x_k^+]) \\ &= [\tilde{h}_1 \otimes 1 + 1 \otimes \tilde{h}_1 - 2x_0^- \otimes x_0^+ , x_k^+ \otimes 1 + 1 \otimes x_k^+ \\ &\quad + \sum_{i=0}^{k-1} h_i \otimes x_{k-i-1}^+ + R] , \end{aligned}$$

where the remainder term $R \in \sum_{p,q,r \geqslant 0} Yx_p^- \otimes Yx_q^+ x_r^+$. Hence, using (1.9),

$$\Delta(x_{k+1}^+) = x_{k+1}^+ \otimes 1 + 1 \otimes x_{k+1}^+ + \sum_{i=0}^{k} h_i \otimes x_{k-i}^+ + R'$$

where

$$\begin{aligned} R' &= \frac{1}{2} [\tilde{h}_1 \otimes 1 + 1 \otimes \tilde{h}_1 - 2x_0^- \otimes x_0^+ , R] - x_0^- \otimes [x_0^+ , x_k^+] \\ &\quad - \sum_{i=0}^{k-1} (h_i x_0^- \otimes [x_0^+ , x_{k-i-1}^+] + 2x_i^- \otimes x_0^+ x_{k-i-1}^+) . \end{aligned}$$

It suffices to check that the first term belongs to $\sum_{p,q,r \geqslant 0} Yx_p^- \otimes Yx_q^+ x_r^+$, and this follows easily from (1.9) and (1.10). This completes the proof.

Finally, we shall need the following analogue of the easy half of the Poincaré-Birkhoff-Witt theorem.

PROPOSITION 1.11. $Y = N^- . H . N^+$.

Proof. The proof is the same as for Lie algebras. Choose any total ordering $\prec$ on the generating set $\{x_k^\pm , h_k\}_{k \in \mathbf{Z}_+}$ such that $x_k^- \prec h_l \prec x_m^+$ for all $k, l, m \in \mathbf{Z}_+$. If $u = u_1 u_2 \dots u_n$ is any monomial in the generators of degree n, define its index

$$\operatorname{ind}(u) = \sum_{i<j} \varepsilon_{ij}$$

where

$$\varepsilon_{ij} = \begin{cases} 0 & \text{if} \quad u_i \prec u_j \\ 1 & \text{if} \quad u_j \prec u_i . \end{cases}$$

Using Lemma 1.7, each monomial can be written as a sum of monomials of smaller degree, or smaller index, and hence, by an obvious induction, as a sum of monomials of index zero.

2. Highest weight representations

By analogy with the definition of highest weight representations of semi-simple Lie algebras, one makes the following

Definition 2.1. A representation V of the Yangian Y is said to be *highest weight* if there is a vector $\Omega \in V$ such that $V = Y\Omega$ and

$$x_k^+ \Omega = 0, \qquad h_k \Omega = d_k \Omega, \qquad k = 0, 1, \dots$$

for some sequence of complex numbers $\mathbf{d} = (d_0, d_1, \dots)$. In this case, Ω is called a highest weight vector of V and $\mathbf{d}$ its highest weight.

Remark. It follows immediately from Definition 1.1 that the assignment $x \mapsto x$ for $x \in \mathfrak{sl}_2$ extends to a homomorphism of algebras $\iota: U(\mathfrak{sl}_2) \to Y$. By Proposition 2.5 below, ι is injective. Thus, any representation of Y can be restricted to give a representation of $\mathfrak{sl}_2$. In particular, we can speak of weights relative to $\mathfrak{sl}_2$ as well as relative to Y. It will always be clear from the context which type of weight is intended.

As in the case of semi-simple Lie algebras, there is a universal highest weight representation of Y of any given highest weight:

Definition 2.2. Let $\mathbf{d} = (d_0, d_1, \dots)$ be any sequence of complex numbers. The *Verma representation* $M(\mathbf{d})$ is the quotient of Y by the left ideal generated by $\{x_k^+, h_k - d_k \cdot 1\}_{k \in \mathbf{Z}_+}$.

PROPOSITION 2.3. *The Verma representation* $M(\mathbf{d})$ *is a highest weight representation with highest weight* $\mathbf{d}$, *and every such representation is* *isomorphic to a quotient of* $M(\mathbf{d})$. *Moreover,* $M(\mathbf{d})$ *has a unique irreducible quotient* $V(\mathbf{d})$.

Proof. Only the last statement requires proof. We consider $M(\mathbf{d})$ as a representation of $\mathfrak{sl}_2$. By Proposition 1.11, the d_0-weight space $\{v \in M(\mathbf{d}) \colon h_0.v = d_0 v\}$ is one-dimensional, and spanned by the highest weight vector $1 \in M(\mathbf{d})$. Thus, if M_1 and M_2 are two proper subrepresentations of $M(\mathbf{d})$, then $M_1 + M_2$ is also proper. It follows that $M(\mathbf{d})$ has a unique maximal proper subrepresentation.

The question of which highest weight representations are finite-dimensional was answered by Drinfel'd in [5, Theorem 2]. His result may be stated as follows.

THEOREM 2.4. *(a) Every irreducible finite-dimensional representation of* Y *is highest weight.*

(b) The irreducible highest weight representation $V(\mathbf{d})$ *of* Y *is finite-dimensional if and only if there exists a monic polynomial* $P \in \mathbf{C}[u]$ *such that*

$$\frac{P(u+1)}{P(u)} = 1 + \sum_{k=0}^{\infty} d_k u^{-k-1} \,,$$

in the sense that the right-hand side is the Laurent expansion of the left-hand side about $u = \infty$.

To construct examples of highest weight representations of Y, we need the following result, which is an immediate consequence of the defining relations (1.1).

PROPOSITION 2.5. *(a) The assignment* $x \mapsto x,\ J(x) \mapsto 0$ *extends to a homomorphism of algebras* $\varepsilon_0 \colon Y \to U(\mathfrak{sl}_2)$.

(b) For any $a \in \mathbf{C}$, *the assignment* $x \mapsto x,\ J(x) \mapsto J(x) + ax$ *extends to an automorphism* τ_a *of* Y.

By part (a), if V is a representation of $\mathfrak{sl}_2$, one can pull it back by ε_0 to give a representation V of Y. Pulling back this representation by τ_a then gives a one-parameter family of representations $V(a)$ of Y. Note that $V(a)$ is an irreducible representation of Y because ε_0 is surjective.

Let W_m be the $(m+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$, $m \in \mathbf{Z}_+$. Then, $W_m(a)$ has a basis $\{e_0, \ldots, e_m\}$ on which the action of Y is given by:

$$x^+.e_i = (i+1)e_{i+1}, \qquad x^-.e_i = (m-i+1)e_{i-1}, \qquad h.e_i = (2i-m)e_i \,,$$

the action of $J(h)$ (resp. $J(x^{\pm})$) being a times that of h (resp. $x^{\pm}$). To make contact with the theory of highest weight representations, we need:

PROPOSITION 2.6. *The action of the generators* $h_k, x_k^{\pm}$ *on* $W_m(a)$ *is given by:*

(1) $x_k^+ . e_i = \left(a - \frac{1}{2}m + i + \frac{1}{2}\right)^k (i+1)e_{i+1}$;

(2) $x_k^- . e_i = \left(a - \frac{1}{2}m + i - \frac{1}{2}\right)^k (m-i+1)e_{i-1}$;

(3) $$h_k . e_i = \left\{\left(a - \frac{1}{2}m + i - \frac{1}{2}\right)^k i(m-i+1) - \left(a - \frac{1}{2}m + i + \frac{1}{2}\right)^k (i+1)(m-i)\right\} e_i .$$

Proof. It is straightforward to check, using the relations (1)-(3) in Theorem 1.2, that these formulas do define a representation of Y. It therefore suffices to check that they also give the correct action of the generators $h, J(h), x^{\pm}, J(x^{\pm})$. This is another straightforward computation, using the isomorphism ϕ in (1.2).

COROLLARY 2.7. *(a)* $W_m(a)$ *is a highest weight representation with highest weight* $\mathbf{d} = (d_0, d_1, \ldots)$ *given by*

$$d_k = m\left(a + \frac{1}{2}m - \frac{1}{2}\right)^k .$$

(b) The monic polynomial P *associated to* $W_m(a)$ *is given by*

$$P(u) = \left(u - a + \frac{1}{2}m - \frac{1}{2}\right)\left(u - a + \frac{1}{2}m - \frac{3}{2}\right) \ldots \left(u - a - \frac{1}{2}m + \frac{1}{2}\right) .$$

Proof. (a) It is clear that e_m is a highest weight vector for $W_m(a)$ relative to Y. The eigenvalues of the h_k on e_m are as stated.

(b) By Theorem 2.4(b), the polynomial P is determined by

$$\frac{P(u+1)}{P(u)} = 1 + \sum_{k=0}^{\infty} m\left(a + \frac{1}{2}m - \frac{1}{2}\right)^k u^{-k-1}$$

$$= \frac{\left(u - a + \frac{1}{2}m + \frac{1}{2}\right)}{\left(u - a - \frac{1}{2}m + \frac{1}{2}\right)} .$$

The stated P clearly satisfies this equation.

In section 4 we shall need to consider the duals of the evaluation representations $W_m(a)$. If V is any finite-dimensional representation of Y, its dual V^* is naturally a representation of Y^{op}, the vector space Y with the opposite multiplication:

$$x.y\,(in\ Y^{op}) = y.x\,(in\ Y) .$$

Moreover, Y^{op} is a Hopf algebra with the same co-multiplication as Y.

PROPOSITION 2.8. *There is an isomorphism of Hopf algebras* $\theta: Y \to Y^{op}$ *such that*

$$\theta(x) = -x , \qquad \theta(J(x)) = J(x)$$

for all $x \in \mathfrak{sl}_2$.

Proof. It is sufficient to prove that the assignment $x \mapsto -x$, $J(x) \mapsto J(x)$ extends to a homomorphism of Hopf algebras $Y \to Y^{op}$. The relations in Y^{op} are obtained by inserting a minus sign on the right-hand side of relations (1) and (3) in (1.1). The result is now clear.

Remark. The anti-homomorphism $\theta: Y \to Y$ is closely related to the antipode S of Y, which is given by

$$S(x) = -x , \qquad S(J(x)) = -J(x) + \frac{1}{4}cx ,$$

where c is the eigenvalue of the Casimir operator in the adjoint representation of $\mathfrak{sl}_2$ (which depends of course on the choice of inner product (,) on $\mathfrak{sl}_2$).

Thus, if V is a finite-dimensional representation of Y, then V^* is a representation of Y with action

$$(y.f)\,(v) = f(\theta(y).v) ,$$

for $y \in Y$, $v \in V$ and $f \in V^*$. Moreover, the fact that θ preserves the co-multiplication implies that $(V_1 \otimes V_2)^* \cong V_1^* \otimes V_2^*$ for any two representations V_1, V_2 of Y.

COROLLARY 2.9. *As representations of* Y, *we have*

$$W_m(a)^* \cong W_m(-a) .$$

Proof. On $W_m(a)$, $J(x)$ acts as ax. Therefore, on $W_m(a)^*$, $J(x)$ acts as $-ax$.

The following is a related result.

PROPOSITION 2.10. *Every evaluation representation* $W_m(a)$ *has a non-degenerate invariant symmetric bilinear form.*

This means that there is a non-degenerate symmetric bilinear form $< , >$ on $W_m(a)$ such that

$$<y.v_1, v_2> = <v_1, \omega(y).v_2> \tag{2.11}$$

for all $y \in Y$, $v_1, v_2 \in W_m(a)$.

Proof. It is well-known that the representation W_m of $\mathfrak{sl}_2$ carries a form $< , >$ which satisfies (2.11) for all $y \in \mathfrak{sl}_2$. Moreover, the form is unique up to a scalar multiple because W_m is irreducible. To prove (2.11) in general, it suffices to check the case $y = x_k^+$, since the case $y = x_k^-$ then follows because $< , >$ is symmetric, and $\omega(x_k^+) = x_k^-$. Since vectors of different weights are orthogonal, it is therefore enough to prove.

$$<x_k^+ . e_i, e_{i+k}> = <e_i, x_k^- . e_{i+k}> \tag{2.12}$$

(with the understanding that $e_i = 0$ unless $0 \leqslant i \leqslant n$). This follows easily from Proposition 2.6 and the invariance of $< , >$ under $\mathfrak{sl}_2$.

3. A COMBINATORIAL INTERLUDE

The form of the polynomial P associated to the representation $W_m(a)$ in Corollary 2.7(b) suggests the following definition.

Definition 3.1. A non-empty finite set of complex numbers is said to be a *string* if it is of the form $\{a, a+1, \ldots, a+n\}$ for some $a \in \mathbf{C}$ and some $n \in \mathbf{N}$. The centre of the string is $a + \frac{n}{2}$ and its length is $n + 1$.

We shall also need:

Definition 3.2. Two strings S_1 and S_2 are said to be *non-interacting* if either

(1) $S_1 \cup S_2$ is not a string, or

(2) $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$.

Remark. We shall discuss the "interactions" of strings in section 4.

We should like to assert that the set of roots of an arbitrary polynomial is a union of non-interacting strings. To make this precise, we need one last definition.

Definition 3.3. A *set with multiplicities* is a map $f:\Sigma \to \mathbf{N}$, where Σ is a set. If Σ is a finite set, the *cardinality* of f is

$$|f| = \sum_{x\in\Sigma} f(x) .$$

The *union* of two sets with multiplicities is the sum of the corresponding maps. Note that any set is a set with multiplicities, all values of the map being equal to one. Also, the roots of a polynomial $P \in \mathbf{C}[u]$ form a set with multiplicities in a natural way. In particular, the roots of the polynomial associated to $W_m(a)$ in Corollary 2.7(b) form a single string

$$S_m(a) = \left\{a - \frac{1}{2}m + \frac{1}{2}, \ldots, a + \frac{1}{2}m - \frac{1}{2}\right\}$$

with centre a and length m.

We shall need the following simple result whose verification we leave to the reader.

LEMMA 3.4. *Two strings $S_m(a)$ and $S_n(b)$ are non-interacting if and only if it is not true that*

$$|a-b| = \frac{1}{2}(m+n), \frac{1}{2}(m+n) - 1, \ldots, \ \textit{or} \ \frac{1}{2}|m-n| + 1 .$$

The result we want is:

PROPOSITION 3.5. *Any finite set of complex numbers with multiplicities can be written uniquely as a union of strings, any two of which are non-interacting.*

Proof. Let $f:\Sigma \to \mathbf{N}$ be a finite set of complex numbers with multiplicities. The proof is by induction on $|f|$. If $|f| = 0$ or 1 there is nothing to prove.

Choose $s \in \Sigma$, let S be the maximal string of numbers in Σ which contains s, and let g be the characteristic function of S. By induction, $f - g$ is a union of non-interacting strings. If T is any such string, then S and T are non-interacting, since if $T \not\subseteq S$ then $S \cup T$ cannot be a string, by maximality of S. Thus, adjoining S to the string decomposition of $f - g$ gives the desired decomposition of f.

As for uniqueness, we first show that the string S above must occur in any decomposition of f as a union of non-interacting strings. For, otherwise, let T be a maximal string in such a decomposition which contains s. Then T is properly contained in S, so there exists $u \in S - T$ such that $T \cup \{u\}$ is a string. Let U be a string in the given decomposition of f which contains u. Then, by its maximality, T cannot be contained in U, so T and U are interacting, a contradiction.

Thus, S must occur in any two decompositions of f as a union of non-interacting strings. Deleting S from both decompositions and using the induction hypothesis, one deduces that the two decompositions are the same.

We conclude this section with the computation of a determinant which plays the same role for Yangians as the Vandermonde determinant plays in the classification of integrable representations of affine Lie algebras [1].

Let r be a positive integer and let b_j, m_j, $1 \leqslant j \leqslant r$, be complex numbers. Quantities $d_{k,j}, A_{k,j}$ for $1 \leqslant j \leqslant r$, $0 \leqslant k \leqslant r-1$, are defined inductively by the following formulas:

$$\begin{aligned} A_{k,j} &= b_j^k + b_j^{k-1} d_{0,j} + \cdots + d_{k-1,j} \\ d_{k,j} &= m_{j+1} A_{k,j+1} + d_{k,j+1}\,, \qquad d_{k,r} = 0 \end{aligned} \tag{3.6}$$

(we set $d_{k,r+1} = 0$). Let A be the matrix $(A_{k,j})$ with $1 \leqslant j \leqslant r$, $0 \leqslant k \leqslant r-1$.

PROPOSITION 3.7. $\det A = \prod_{1 \leqslant k < j \leqslant r} (b_j - b_k - m_j)$.

Remark. One can think of $\det A$ as a "quantum Vandermonde determinant". Indeed, recall that Y is obtained from a deformation of $U(\mathfrak{sl}_2[t])$ by setting the deformation parameter h equal to one. If we had not set $h = 1$, then in equation (3.6) $d_{k,j}$ would be replaced by $hd_{k,j}$ and in equation (3.7) m_j would be replaced by hm_j. Thus, in the "classical limit" $h \to 0$, $\det A$ becomes the usual Vandermonde determinant and (3.7) its well-known factorization.

Our proof of (3.7) is rather indirect and will be given in the next section.

4. CLASSIFICATION

To construct further examples of finite-dimensional representations of Y, we consider tensor products of the evaluation representations $W_m(a)$. In general, if W_1 and W_2 are two representations of Y, the action of Y on the tensor product is given by

$$x.(w_1 \otimes w_2) = \Delta(x)\,(w_1 \otimes w_2)\;,$$

the action of the right-hand side being that of $Y \otimes Y$ on $W_1 \otimes W_2$. More generally, an r-fold tensor product $W_1 \otimes \cdots \otimes W_r$ is defined using the homomorphism $\Delta^{(r)}\colon Y \to \underbrace{Y \otimes \cdots \otimes Y}_{r}$ given by

$$\Delta^{(r)} = (\Delta \otimes id \otimes \cdots \otimes id)\Delta^{(r-1)}\;, \qquad \Delta^{(2)} = \Delta\;.$$

Note that, since Δ is co-associative, an equivalent inductive definition is:

$$x.(w_1 \otimes w_2 \otimes \cdots \otimes w_r) = \Delta(x)\,\big(w_1 \otimes (w_2 \otimes \cdots \otimes w_r)\big)\;.$$

Our first main result can now be stated as follows.

THEOREM 4.1. *A tensor product* $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ *is an irreducible representation of* Y *if and only if the strings* $S_{m_i}(a_i)$ *are in general position.*

The proof is in several steps. We begin by analyzing the tensor product $W_m(a) \otimes W_n(b)$ of two evaluation representations. Recall that, as representations of $\mathfrak{sl}_2$, we have

$$W_m(a) \otimes W_n(b) \cong W_{m+n} \oplus W_{m+n-2} \oplus \cdots \oplus W_{|m-n|}\;.$$

We shall refer to the copy of W_{m+n} inside $W_m(a) \otimes W_n(b)$ as its highest component.

The following result proves Theorem 4.1 in the case $r = 2$.

PROPOSITION 4.2. *(a) The tensor product* $W_m(a) \otimes W_n(b)$ *has a proper Y-subrepresentation not containing the highest component if and only if*

$$a - b = \frac{1}{2}(m+n) - p + 1$$

for some $0 < p \leqslant \min\{m, n\}$.

(b) The highest component generates a proper Y-subrepresentation of $W_m(a) \otimes W_n(b)$ *if and only if*

$$b - a = \frac{1}{2}(m+n) - p + 1$$

for some $0 < p \leqslant \min\{m, n\}$.

We need two preliminary lemmas. Let Ω_q denote a highest weight vector (for $\mathfrak{sl}_2$) in the component W_{m+n-2q} of $W_m(a) \otimes W_n(b)$.

LEMMA 4.3. *If* $a - b = \frac{1}{2}(m+n) - p + 1$ *for some* $0 < p \leqslant \min\{m, n\}$, *then*

$$J(h).\Omega_p \in \operatorname{span}\{\Omega_p\} ;$$
$$J(x^+).\Omega_p = 0 ;$$
$$J(x^-).\Omega_p \in \operatorname{span}\{\Omega_{p+1}, x^-.\Omega_p\} .$$

Proof. The vector Ω_p is given by

$$\Omega_p = \sum_{i=0}^{p} (-1)^i \frac{(m-i)!(n-p+i)!}{m!(n-p)!} e_{m-i} \otimes e_{n-p+i} .$$

(To verify this, it is enough to check that Ω_p has the correct weight and that $x^+.\Omega_p = 0$. We omit the simple computation.) From (1.1) we find

$$\Delta(J(x^+)) = J(x^+) \otimes 1 + 1 \otimes J(x^+) - \frac{1}{2} x^+ \otimes h + \frac{1}{2} h \otimes x^+.$$

Hence,

$$J(x^+).\Omega_p$$
$$= \sum_{i=0}^{p} (-1)^i \frac{(m-i)!(n-p+i)!}{m!(n-p)!}\left(\left(a - \frac{1}{2}(n-2p+2i)\right)(m-i+1)e_{m-i+1} \otimes e_{n-p+i}\right.$$
$$\left. + \left(b + \frac{1}{2}(m-2i)\right)(n-p+i+1)e_{m-i} \otimes e_{n-p+i+1}\right).$$

The coefficient of $e_{m-i} \otimes e_{n-p+i+1}$ is

$$(-1)^i \frac{(m-i)!(n-p+i)!}{m!(n-p)!}\left(b + \frac{1}{2}(m-2i)\right)(n-p+i+1)$$

$$+(-1)^{i+1}\frac{(m-i-1)!\,(n-p+i+1)!}{m!\,(n-p)!}\left(a-\frac{1}{2}(n-2p+2i+2)\right)(m-i)$$

$$=(-1)^{i}\frac{(m-i-1)!\,(n-p+i)!}{m!\,(n-p)!}(m-i)(n-p+i+1)\left(b-a+\frac{1}{2}(m+n)-p+1\right),$$

which is zero by our assumption on $a-b$.

The proof of the statements involving $J(h)$ and $J(x^-)$ is similar. We omit the details.

Similar arguments prove the second lemma. Again, we shall omit the details.

LEMMA 4.4. *For any* $0 \leqslant q \leqslant \min\{m, n\}$, *we have*

$$J(h).\Omega_q \in \operatorname{span}\{\Omega_q, x^-.\Omega_{q-1}\};$$
$$J(x^+).\Omega_q \in \operatorname{span}\{\Omega_{q-1}\};$$
$$J(x^-).\Omega_q \in \operatorname{span}\{\Omega_{q+1}, x^-.\Omega_q, (x^-)^2.\Omega_{q-1}\}.$$

Proof of Proposition 4.2.

(a) Suppose that $a-b=\frac{1}{2}(m+n)-p+1$ for some $0 < p \leqslant \min\{m, n\}$.

We shall prove that

$$V = W_{m+n-2p} \oplus \cdots \oplus W_{|m-n|}$$

is a Y-subrepresentation of $W_m(a) \otimes W_n(b)$. It is enough to show that $(x^-)^r.\Omega_q \in V$ if $p \leqslant q \leqslant \min\{m, n\}$ and $0 \leqslant r \leqslant m+n-2q$. We prove this by induction on r. If $r=0$ there is nothing to prove. For any $r \geqslant 1$, we have

$$J(h).(x^-)^r.\Omega_q = -2J(x^-).(x^-)^{r-1}.\Omega_q + x^-.J(h).(x^-)^{r-1}.\Omega_q;$$
$$J(x^+).(x^-)^r.\Omega_q = J(h).(x^-)^{r-1}.\Omega_q + x^-.J(x^+).(x^-)^{r-1}.\Omega_q;$$
$$J(x^-).(x^-)^r.\Omega_q = (x^-)^r.J(x^-).\Omega_q.$$

The induction hypothesis, together with Lemmas 4.3 and 4.4, shows that the right-hand sides of these formulas are elements of V.

For the converse, suppose that V is a proper subrepresentation of $W_m(a) \otimes W_n(b)$ which does not contain the highest component. Then, for some $0 < p \leqslant \min\{m, n\}$, we shall have $\Omega_p \in V$ but $\Omega_q \notin V$ if $q < p$. Then, $J(x^+).\Omega_p = 0$, and by the computation in the proof of Lemma 4.3, this implies that $a-b=\frac{1}{2}(m+n)-p+1$.

(b) We shall deduce the second part of the Proposition from the first part using duality. By Corollary 2.9, we have

$$(W_m(a) \otimes W_n(b))^* \cong W_m(-a) \otimes W_n(-b) .$$

Hence, V is a proper subrepresentation of $W_m(a) \otimes W_n(b)$ containing the highest component if and only if the annihilator V^o of V is a proper subrepresentation of $W_m(-a) \otimes W_n(-b)$ *not* containing the highest component. By part (a), $W_m(-a) \otimes W_n(-b)$ has such a subrepresentation if and only if $b - a = \frac{1}{2}(m+n) - p + 1$ for some $0 < p \leqslant \min\{m, n\}$.

Proposition 4.2 can be made more precise.

PROPOSITION 4.5. *Let* $W = W_m(a) \otimes W_n(b), 0 < p \leqslant \min\{m, n\}$. *If* $| a - b | = \frac{1}{2}(m+n) - p + 1$, *then* W *has a unique proper subrepresentation* V. *In fact:*

(a) if $a - b = \frac{1}{2}(m+n) - p + 1$, *we have*

$$V \cong W_{m-p}\left(a + \frac{1}{2}p\right) \otimes W_{n-p}\left(b - \frac{1}{2}p\right) ,$$

$$W/V \cong W_{p-1}\left(a - \frac{1}{2}(m-p+1)\right) \otimes W_{m+n-p+1}\left(b + \frac{1}{2}(m-p+1)\right) ,$$

and, as a representation of $\mathfrak{sl}_2$.

$$V \cong W_{m+n-2p} \oplus \cdots \oplus W_{|m-n|} ;$$

(b) if $b - a = \frac{1}{2}(m+n) - p + 1$, *then*

$$V \cong W_{p-1}\left(a + \frac{1}{2}(m-p+1)\right) \otimes W_{m+n-p+1}\left(b - \frac{1}{2}(m-p+1)\right) ,$$

$$W/V \cong W_{m-p}\left(a - \frac{1}{2}p\right) \otimes W_{n-p}\left(b + \frac{1}{2}p\right) ,$$

and, as a representation of $\mathfrak{sl}_2$,

$$V \cong W_{m+n} \oplus \cdots \oplus W_{m+n-2p+2} .$$

The proof of Proposition 4.2 already gives the uniqueness statements and the isomorphism type under $\mathfrak{sl}_2$. The determination of V as a representation of Y is made using Proposition 1.6 and Theorem 2.4. Since we shall not use this result in the proof of the classification theorem, we omit the details.

Note that Proposition 4.5 (in conjunction with Corollary 4.7 below) enables one to determine the composition series of any tensor product of evaluation representations.

Proposition 4.5 has an interesting string-theoretic interpretation. In (4.5)(a), the subrepresentation corresponds to the "annihilation" of the two strings $S_m(a)$ and $S_n(b)$: the intersection of the strings, together with the two nearest neighbour elements, is discarded, leaving two new strings (in exceptional cases, only one string might remain, or the strings might even annihilate each other completely). Note that the two new strings are always non-interacting. The annihilation interaction is illustrated in the following diagram.

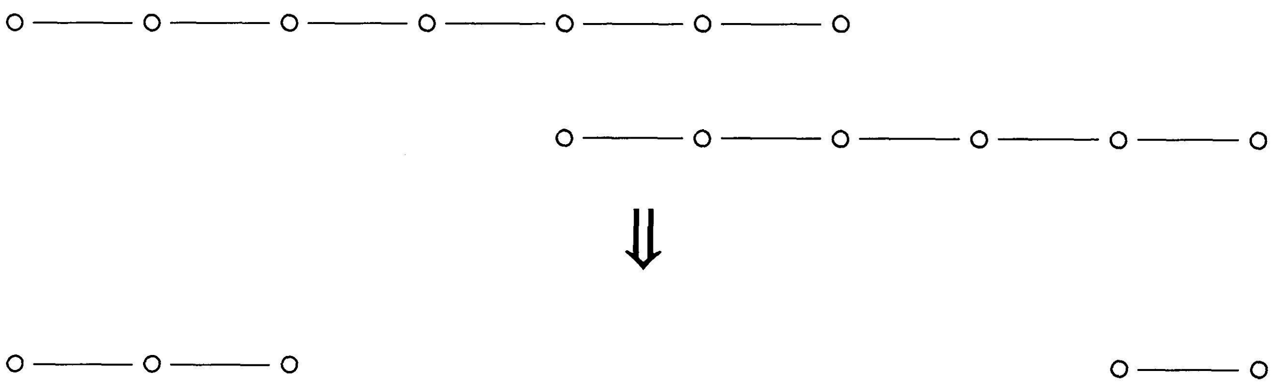

FIGURE 1:
Annihilation of two strings.

The quotient representation in (4.5)(a) corresponds to the "fusion" of the two strings $S_m(a)$ and $S_n(b)$: the two new strings produced by this operation are those which form the unique decomposition of $S_1 \cup S_2$ into the sum of two non-interacting strings (in exceptional cases, only one new string is produced). The fusion interaction is illustrated in the following diagram.

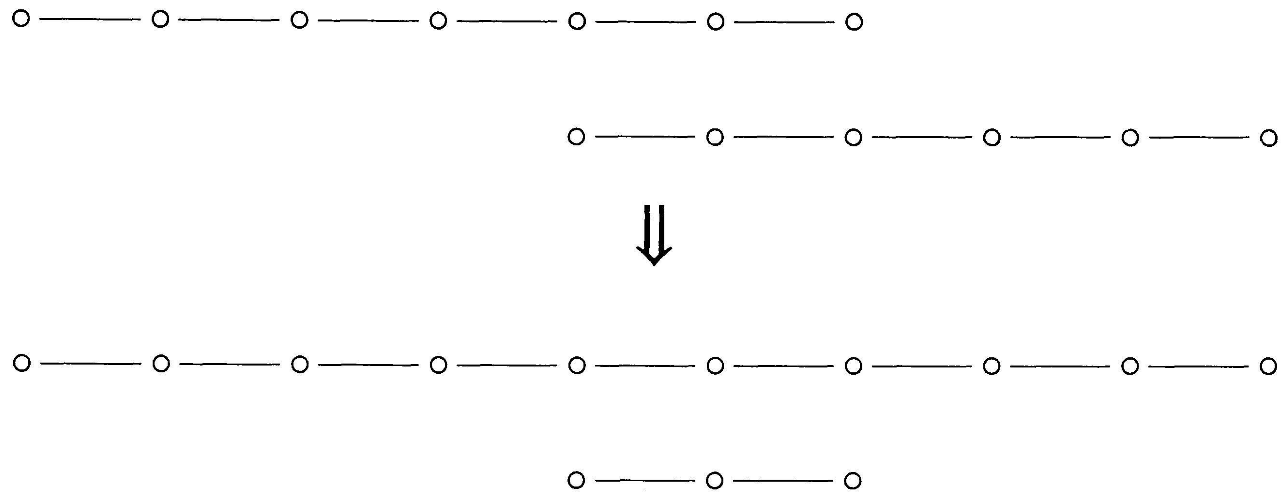

FIGURE 2:
Fusion of two strings.

In (4.5)(b), the roles of the two strings are reversed, and the subrepresentation corresponds to the fusion of the two strings and the quotient to their annihilation.

We now move on to consider tensor products of an arbitrary number of evaluation representations. We begin with:

PROPOSITION 4.6. *If* $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ *is irreducible, then it is highest weight and the polynomial associated to it by Theorem 2.4(b) is the product of the polynomials associated to each factor in the tensor product.*

Proof. It follows from Proposition 1.6(2) that the tensor product of the highest weight vectors in the $W_{m_i}(a_i)$ is a highest weight vector in the tensor product.

As for the second statement, by an easy induction argument using (1.6)(1) and (1.6)(2), we find that

$$\Delta^{(r)}(h_k) = \sum h_{k_1} h_{k_2} \ldots h_{k_r} \text{ modulo } \sum\nolimits_{p \geqslant 0} Y \otimes Yx_p^+ + Yx_p^+ \otimes Y$$

where the first sum is over all r-tuples $k_1, k_2, \ldots, k_r$ such that $k_i \geqslant -1$ and $\sum k_i = k - r + 1$ (h_{-1} is interpreted as the identity element 1). Hence, the eigenvalue of h_k on the highest weight vector in $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ is, by Proposition 2.6(a),

$$d_k = \sum_{\{k_i\}} \prod_{i=1}^{r} m_i \left(a + \frac{1}{2} m_i - \frac{1}{2} \right)^{k_i}.$$

It is easy to see that this is equal to the coefficient of u^{-k-1} in the product

$$\prod_{i=1}^{r} \left(1 + \sum_{k_i=0}^{\infty} m_i \left(a + \frac{1}{2} m_i - \frac{1}{2} \right)^{k_i} u^{-k_i-1} \right) = \prod_{i=1}^{r} \frac{P_i(u+1)}{P_i(u)},$$

where P_i is the polynomial associated to the representation $W_{m_i}(a_i)$. This completes the proof.

COROLLARY 4.7. *If* $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ *is irreducible, then it is unchanged, up to isomorphism, by any permutation of the factors in the tensor product.*

Proof. Let $V = \bigotimes_{i=1}^{r} W_{m_i}(a_i)$ and let V' be the result of applying some permutation to the factors in the tensor product. Applying the same permutation to the highest weight vector in V gives a highest weight vector in V' of the same weight. It follows from Proposition 2.3 that V is isomorphic to a subquotient of V'. Since V and V' have the same dimension, they must be isomorphic.

Remark. It is *not* true that the permutation of the factors is an isomorphism $V \cong V'$ of representations of Y.

We can now prove the "only if" half of Theorem 4.1. Suppose that some pair of strings $S_{m_j}(a_j)$ and $S_{m_k}(a_k)$ are interacting. Then, by Corollary 4.7, $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ is isomorphic to a tensor product in which $S_{m_j}(a_j)$ and $S_{m_k}(a_k)$ are adjacent. By Proposition 4.2, the latter representation is reducible.

We now turn to the proof of the "if" part of Theorem 4.1. Note first that there is no loss of generality in assuming that $m_1 \leqslant \cdots \leqslant m_r$. Indeed, since the strings $S_{m_i}(a_i)$ are assumed to be non-interacting, it follows from (4.2) and (4.7) that the tensor product of any pair of the evaluation representations $W_{m_i}(a_i)$ is unchanged, up to isomorphism, by an interchange of the two two factors. Hence, $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ is unchanged, up to isomorphism, by any permutation of its factors, since the permutation can be effected by a sequence of interchanges of nearest neighbours.

We shall assume that $m_1 \leqslant \cdots \leqslant m_r$ for the rest of the proof of (4.1). The main step in the proof is the following result.

PROPOSITION 4.8. *Suppose that the strings* $S_{m_i}(a_i)$, $1 \leqslant i \leqslant r$, *are non-interacting, and that* $m_1 \leqslant \cdots \leqslant m_r$. *Then* $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ *is generated by the tensor product of the highest weight vectors in the* $W_{m_i}(a_i)$.

Assuming this result for a moment, the proof of Theorem 4.1 is completed as follows. Suppose that the strings $S_{m_i}(a_i)$ are non-interacting. Note that, as a representation of $\mathfrak{sl}_2$, $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ contains a unique highest component $W_M, M = \sum m_i$. By (4.2), (4.7) and (4.8), if $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ has a proper subrepresentation V, then V does not contain W_M. But then the annihilator V^o of V is a proper subrepresentation of the dual

$$(\bigotimes_{i=0}^{r} W_{m_i}(a_i))^* \cong \bigotimes_{i=0}^{r} W_{m_i}(-a_i)$$

which does contain its highest component. By (4.2), (4.7) and (4.8) again, this is impossible.

Remark. The following is an interesting alternative argument. By Proposition 2.10, each factor $W_{m_i}(a_i)$ has an invariant bilinear form. If W_1 and W_2 are two representations of Y which have invariant forms $< \, , \, >_1$ and $< \, , \, >_2$, then there is an invariant bilinear map

$$(W_1 \otimes W_2) \times (W_2 \otimes W_1) \to \mathbf{C}$$

given by

$$< w_1 \otimes w_2, w_2' \otimes w_1' > \; = \; < w_1, w_1' >_1 < w_2, w_2' >_2 .$$

(The change of order is necessary because Y is not co-commutative.) In particular, if $W_1 \otimes W_2 \cong W_2 \otimes W_1$, then $W_1 \otimes W_2$ has an invariant bilinear form. Using this observation, the fact that $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ is unchanged, up to isomorphism, by any permutation of the factors (which follows from (4.2) and (4.7)), and an easy induction, one sees that $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ has an invariant bilinear form. But now a standard argument in the theory of Lie algebras shows that a highest weight representation which carries a non-zero invariant bilinear form is irreducible.

We must now give the

Proof of Proposition 4.8. By induction on r. The result is known if $r = 1$ or 2.

We first prove that $W = \bigotimes_{i=1}^{r} W_{n_i}(a_i)$ is generated by the vector $e_0 \otimes \Omega'$, where $\Omega' = e_{m_2} \otimes \cdots \otimes e_{m_r}$ is the highest weight vector in $W' = \bigotimes_{i=2}^{r} W_{m_i}(a_i)$. By the induction hypothesis, W' is generated by Ω'. From Proposition 1.11, for any $w' \in W'$, there exists $y \in N^-$ such that $y.\Omega' = w'$. Then

$$y.(e_0 \otimes \Omega') = \Delta(y)\,(e_0 \otimes \Omega') = e_0 \otimes w' ,$$

where the last equality follows from Proposition 1.6(3) and the fact that $x_k^- . e_0 = 0$. Hence, $e_0 \otimes W' \subseteq Y.(e_0 \otimes \Omega')$. Now an easy induction on i proves that $e_i \otimes W' \subseteq Y.(e_0 \otimes \Omega')$ for $0 \leqslant i \leqslant m_1$: for the inductive step one uses the fact that

$$e_{i+1} \otimes W' = x^+ . e_i \otimes W' \subseteq e_i \otimes W' + x^+ . (e_i \otimes W') .$$

This proves our assertion.

We now prove that $e_0 \otimes \Omega' \in Y.\Omega$, where $\Omega = e_{m_1} \otimes \cdots \otimes e_{m_r}$. For any $i > 0$, consider the equations

$$(4.9) \qquad x_k^- . (e_i \otimes \Omega') = \Big(\sum_{p=0}^{k} b_1^p d_{k-p-1,1} x^- . e_i\Big) \otimes \Omega' + e_i \otimes x_k^- . \Omega' ,$$

for $k = 0, \ldots, r-1$, where $b_1 = a_1 - \frac{1}{2} m_1 + i - \frac{1}{2}$, $d_{k,1}$ is the eigenvalue of h_k on Ω' (and $d_{-1,1} = 0$). Equation (4.9) follows from Proposition 1.6 (3), using the fact that Ω' is a highest weight vector for Y. More generally, iterating (4.9), we find that

$$(4.10) \qquad x_k^- . (e_i \otimes \Omega') = \sum_{j=1}^{r} A_{k,j} e_i \otimes \cdots \otimes x^- . e_{m_j} \otimes \cdots \otimes e_{m_r} ,$$

where

$$A_{k,j} = \sum_{p=0}^{k} b_j^p d_{k-p-1,j} ,$$

$$b_j = a_j + \frac{1}{2} m_j - \frac{1}{2} \quad \text{for} \quad j \geqslant 2 ,$$

and $d_{k,j}$ is the eigenvalue of h_k on $e_{m_{j+1}} \otimes \cdots \otimes e_{m_r}$ (and $d_{-1,j} = 1$).

Using Proposition 1.6(1), one sees that

$$d_{k,j} = m_{j+1} A_{k,j+1} + d_{k,j+1}$$

so we are in the situation of (3.6). Assuming Proposition 3.7, which has yet to be proved, we have

$$\det A = \prod_{1 \leqslant k < j \leqslant r} (b_j - b_k - m_j) .$$

Since the strings $S_{m_j}(a_j)$ are non-interacting, this determinant is non-zero. For, $b_j = b_k + m_j$ for some $j > k > 1$ implies that $a_j - a_k = \frac{1}{2}(m_j + m_k)$, which is impossible; and $b_j = b_1 + m_j$ implies that $a_j - a_1 = \frac{1}{2}(m_j - m_1) - i = \frac{1}{2}|m_j - m_1| - i$, which is also impossible since $i > 0$.

Hence, equation (4.10) implies that e_{i-1} is a linear combination of the elements $x_k^- .(e_i \otimes \Omega')$ for $0 \leqslant k \leqslant r - 1$. An obvious (downward) induction now proves that $e_i \otimes \Omega' \in Y.(e_{m_1} \otimes \Omega') = Y.\Omega$ for all $i \geqslant 0$. In particular, we have proved that $e_0 \otimes \Omega' \in Y.\Omega$.

All that remains is to prove Proposition 3.7. We show first that $b_j - b_{j-1} - m_j$ is a root of $\det A$ for $2 \leqslant j \leqslant r$. In fact, we shall prove that, if $b_j - b_{j-1} - m_j = 0$, then the j-th and $(j+1)$-th columns of the matrix A are the same. To begin with, $A_{0,j} = A_{0,j-1} = 1$. Proceeding by induction on k and using (3.6), we have

$$\begin{aligned} A_{k+1,j-1} &= b_{j-1}A_{k,j-1} + d_{k,j-1} \\ &= b_{j-1}A_{k,j} + d_{k,j-1} \\ &= (b_j - m_j)A_{k,j} + d_{k,j-1} \\ &= (b_j - m_j)A_{k,j} + m_jA_{k,j} + d_{k,j} \\ &= b_jA_{k,j} + d_{k,j} \\ &= A_{k+1,j} , \end{aligned}$$

which proves our assertion.

If $j > k$ is any pair of indices, there is a permutation σ of $\{1, \dots, r\}$ such that $\sigma(1) = 1$ and $\sigma(k) = \sigma(j) - 1$. Let Ω'_σ be the result of applying σ to the factors in Ω', and define W_σ and W'_σ similarly. As we remarked earlier, W' and W'_σ are isomorphic as representations of Y, and the isomorphism must preserve highest weight vectors. Hence, there is an isomorphism $W \cong W_\sigma$ which takes $e_i \otimes \Omega'$ to $e_i \otimes \Omega'_\sigma$ for all i. Hence,

$$\{x_0^- .(e_i \otimes \Omega'), \dots, x_{r-1}^- .(e_i \otimes \Omega')\}$$

is linearly dependent if and only if

$$\{x_0^- .(e_i \otimes \Omega'_\sigma), \dots, x_{r-1}^- .(e_i \otimes \Omega'_\sigma)\}$$

is linearly dependent. By (4.10), the first condition holds if and only if $\det A = 0$, and the second if and only if $\det A_\sigma = 0$, where A_σ is the matrix obtained by applying σ to the parameters $a_1, \ldots, a_r, m_1, \ldots, m_r$. This implies that $b_j - b_k - m_j$ is a root of $\det A$ if and only if $b_{\sigma(j)} - b_{\sigma(k)} - m_{\sigma(j)}$ is a root of $\det A_\sigma$, and this is true by the first part of the argument.

We have now proved that $b_j - b_k - m_j$ is a root of $\det A$ for all $j > k$. This proves Proposition 3.7 in the case of interest to us, namely when the m_j are positive integers. But since (3.7) is a polynomial identity, it holds in general.

The proof of Theorem 4.1 is now complete.

The following result completes the classification of the finite-dimensional irreducible representations of Y.

THEOREM 4.11. *(a) Every finite-dimensional irreducible representation of Y is isomorphic to a tensor product of evaluation representations $W_m(a)$.*

(b) Two irreducible tensor products of evaluation representations are isomorphic as representations of Y if and only if one is obtained from the other by a permutation of the factors in the tensor product.

Proof. (a) Let V be a finite-dimensional irreducible representation of Y. Let P be the polynomial corresponding to V in Theorem 2.4. The roots of P form a set with multiplicities which, by (3.5), can be written as a union of non-interacting strings. Let $S_{m_i}(a_i)$ be the strings which occur (the m_i, a_i are not necessarily distinct). By (4.1) and (4.6), the tensor product $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ is irreducible and has P as its associated polynomial (by (4.7), the order of the factors in the tensor product is immaterial). By Theorem 2.4, V is isomorphic to $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$.

(b) Suppose that

$$\otimes\, W_{m_i}(a_i) \cong \otimes\, W_{n_j}(b_j)\ .$$

are irreducible representations of Y. Then, both tensor products are associated to the same polynomial P. The $S_{m_i}(a_i)$ and the $S_{n_j}(b_j)$ both give decompositions of the roots of P into sets of non-interacting strings. By Proposition 3.5, the decompositions are the same. This means that the factors $W_{m_i}(a_i)$ and $W_{n_j}(b_j)$ are the same up to a permutation.

5. R-MATRICES AND INTERTWINING OPERATORS

In this section we shall prove that, after a trivial twisting, the intertwining operators between certain representations of Yangians provide rational solutions of the quantum Yang-Baxter equation. Recall that, if V is any representation of $Y = Y(\mathfrak{sl}_2)$, then, for any $a \in \mathbf{C}$, we denote by $V(a)$ its pull-back by the automorphism τ_a of Y defined in Proposition 2.5.

PROPOSITION 5.1. *Let V, W be irreducible finite-dimensional representations of Y with highest weight vectors Ω_V, Ω_W and let $a, b \in \mathbf{C}$. Then:*

(a) the tensor products $V(a) \otimes W(b)$ and $W(b) \otimes V(a)$ are irreducible and isomorphic except for a finite set of values $S(V, W)$ of $a - b$;

(b) the unique intertwining operator

$$I(V, a; W, b) \colon W(b) \otimes V(a) \to V(a) \otimes W(b)$$

which maps $\Omega_W \otimes \Omega_V$ to $\Omega_V \otimes \Omega_W$ is a rational function of $a - b$ with values in $\operatorname{Hom}(W \otimes V, V \otimes W)$.

Proof. Part (a) follows immediately from Proposition 4.2 and Corollary 4.7. For part (b), we need the following lemma.

LEMMA 5.2. *Let V, W be representations of Y and let $a \in \mathbf{C}$.*

(a) If V is irreducible, so is $V(a)$.

(b) If $I \colon V \to W$ is an isomorphism of representations of Y, so is $I \colon V(a) \to W(a)$.

Proof of lemma. Part (a) follows from the definition of $V(a)$. For part (b), we must show that I commutes with the action of x and $J(x)$ on $V(a)$ and $W(a)$, for all $x \in \mathfrak{sl}_2$. But this is clear, since the action of x is the same as that on V and W, and that of $J(x)$ is the same as that of $J(x) + ax$ on V and W.

Returning to the proof of Proposition 5.1, it follows from the lemma that $I(V, a; W, b)$ is a function of $a - b$, so it suffices to consider the case $b = 0$. For any $a \in \mathbf{C}$ which does not belong to the finite set $S(V, W)$, there is a unique isomorphism

$$I(V, a; W, 0) \equiv I(a) \colon W \otimes V(a) \to V(a) \otimes W$$

of representations of Y such that

$$I(a)\,(\Omega_W \otimes \Omega_V) = \Omega_V \otimes \Omega_W\,. \tag{5.3}$$

Choose bases of $V \otimes W$ and $W \otimes V$ and let $\{I_\lambda\}$ be a basis of $\mathfrak{sl}_2$; write $I(a)$ also for the matrix of $I(a)$ with respect to these bases. Let A_λ, B_λ be the matrices of I_λ and $J(I_\lambda)$ acting on $W \otimes V(a)$; and let A'_λ and B'_λ refer similarly to $V(a) \otimes W$. Then, $I(a)$ commutes with the action of Y if and only if $I(a)$ satisfies the following system of homogeneous linear equations:

$$A_\lambda I(a) = I(a)A'_\lambda, \quad B_\lambda I(a) = I(a)B'_\lambda, \quad \text{for all} \quad \lambda .$$

We know that, if $a \notin S(V, W)$, these equations have a unique solution satisfying equation (5.3). By elementary linear algebra, the solution is a rational function of the entries of the matrices $A_\lambda, A'_\lambda, B_\lambda, B'_\lambda$. Since A_λ, A'_λ are independent of a and B_λ, B'_λ are linear in a, the result follows.

Definition 5.4. Let V be a finite-dimensional irreducible representation of Y. Then, the *R-matrix associated to* V is the function $R(a-b)$ with values in $\mathrm{End}(V \otimes V)$ given by

$$R(a-b) = I(V, a; V, b)\sigma ,$$

where $\sigma \in \mathrm{End}(V \otimes V)$ is the switch of the two factors.

THEOREM 5.5. *Let V be a finite-dimensional irreducible representation of Y. Then the R-matrix associated to V is a rational solution of the quantum Yang-Baxter equation:*

$$R^{12}(a-b)R^{13}(a-c)R^{23}(b-c) = R^{23}(b-c)R^{13}(a-c)R^{12}(a-b) . \tag{5.6}$$

Proof. We note first some simple commutation relations between the intertwining operator $I(a-b) \equiv I(V, a; V, b)$ and the switch map σ. For example, we have

$$\sigma^{12} I^{13}(a-c)\sigma^{12} = I^{23}(a-c) .$$

by an easy computation. Similarly,

$$\sigma^{12}\sigma^{13} I^{23}(b-c)\sigma^{13}\sigma^{12} = I^{12}(b-c) .$$

Hence,

$$\begin{aligned} R^{12}(a-b)R^{13}(a-c)R^{23}(b-c) &= I^{12}(a-b)\sigma^{12} I^{13}(a-c)\sigma^{13} I^{23}(b-c)\sigma^{23} \\ &= I^{12}(a-b)I^{23}(a-c)\sigma^{12}\sigma^{13} I^{23}(b-c)\sigma^{23} \\ &= I^{12}(a-b)I^{23}(a-c)I^{12}(b-c)\sigma^{12}\sigma^{13}\sigma^{23} . \end{aligned}$$

Similarly,

$$R^{23}(b-c)R^{13}(a-c)R^{12}(a-b) = I^{23}(b-c)I^{12}(a-c)I^{23}(a-b)\sigma^{23}\sigma^{13}\sigma^{12} .$$

Hence, in view of the relation

$$\sigma^{12}\sigma^{13}\sigma^{23} = \sigma^{23}\sigma^{13}\sigma^{12}$$

in the symmetric group on three letters, the equation to be proved is

(5.7) $$I^{12}(a-b)I^{23}(a-c)I^{12}(b-c) = I^{23}(b-c)I^{12}(a-c)I^{23}(a-b) .$$

Note that both sides of equation (5.7) define intertwining operators

$$V(c) \otimes V(b) \otimes V(a) \to V(a) \otimes V(b) \otimes V(c)$$

which fix the tensor product of the highest weight vectors in V. Hence, regarded as functions on $\mathbf{C}^3$ with values in $\mathrm{End}(V \otimes V \otimes V)$, they agree on the complement of the set S of $(a, b, c) \in \mathbf{C}^3$ where $V(c) \otimes V(b) \otimes V(a)$ or $V(a) \otimes V(b) \otimes V(c)$ is reducible. It follows from part (a) of Proposition 5.1 that S intersects each complex line parallel to one of the axes in $\mathbf{C}^3$ in at most finitely many points. It is easy to see that the complement of such a set is Zariski dense in $\mathbf{C}^3$. Since the two sides of equation (5.7) are rational functions which agree on a Zariski dense set, they are equal.

Remark. We have used the following simple fact about intertwining operators. Let U, V and W be representations of a Yangian $Y(\mathfrak{sl}_2)$ and let $I: U \otimes V \to V \otimes U$ be an intertwining operator. Then

$$I^{12}: U \otimes V \otimes W \to V \otimes U \otimes W$$

and

$$I^{23}: W \otimes U \otimes V \to W \otimes V \otimes U$$

are intertwining operators. While this is obvious enough, it should be noted that

$$I^{13}: U \otimes W \otimes V \to V \otimes W \otimes U$$

is *not* an intertwining operator in general.

We conclude this general discussion by showing that, up to a sign change in the parameter, the R-matrix $R(u)$ we have associated to a representation of Y is the same as that constructed using the "universal R-matrix" (see Theorem 3 of [4]). Set

$$\tilde{R}(u) = R(-u) .$$

Then, by Theorem 4 of [4], it suffices to prove that

(5.8) $$P_\lambda^+(a, b)\tilde{R}(b-a) = \tilde{R}(b-a)P_\lambda^-(a, b)$$

where

$$P_\lambda^{\pm}(a, b) = (\rho \otimes \rho)\,((J(I_\lambda) + aI_\lambda) \otimes 1 + 1 \otimes (J(I_\lambda) + bI_\lambda) + \frac{1}{2}[I_\lambda \otimes 1, \Omega])\,,$$

$\rho : Y \to \mathrm{End}(V)$ is the action of Y on V and $\{I_\lambda\}$ is an orthonormal basis of $\mathfrak{sl}_2$. In terms of intertwining operators, equation (5.8) asserts that

$$P_\lambda^{+}(a, b) I(a-b) = I(a-b)\sigma P_\lambda^{-}(a, b)\sigma\,.$$

But it is easy to see that

$$\sigma P_\lambda^{-}(a, b)\sigma = P_\lambda^{+}(b, a)\,.$$

Hence, we must prove that

$$P_\lambda^{+}(a, b) I(a-b) = I(a-b) P_\lambda^{+}(b, a)\,.$$

But this is simply the statement that

$$I(a-b) : V(b) \otimes V(a) \to V(a) \otimes V(b)$$

commutes with the action of $J(I_\lambda)$.

We shall now apply these results to compute the R-matrices associated to every finite-dimensional irreducible representation of Y. By Theorem 4.11, every such representation is of the form

$$V = V_{m_1}(a_1) \otimes \cdots \otimes V_{m_k}(a_k).$$

The intertwining operator

$$I(a-b) : V(b) \otimes V(a) \to V(a) \otimes V(b)$$

can be computed as the product of k^2 intertwining operators of the form $I(V_m, a; V_n, b)$, each of which effects an interchange of nearest neighbours. Since such an operator commutes, in particular, with the action of $\mathfrak{sl}_2$, it can be written in the form

$$(5.9) \qquad I(V_m, a; V_n, b) = \sum_{j=0}^{\min\{m, n\}} c_j P_{m+n-2j}\,,$$

where

$$P_{m+n-2j} : V_n \otimes V_m \to V_m \otimes V_n$$

is the projection onto the irreducible component of

$$V_m \otimes V_n \cong \otimes_{j=0}^{\min\{m, n\}} V_{m+n-2j}$$

of type V_{m+n-2j}. We have $c_0 = 1$ since $I(V_m, a; V_n, b)$ preserves the tensor products of the highest weight vectors.

To compute $I(V_m, a; V_n, b)$, let $\Omega_j, j = 0, 1, \ldots, \min\{m, n\}$, be a highest weight vector in $V_n \otimes V_m$ of weight $m + n - 2j$; then, the vector Ω_j' obtained by switching the order of the factors in Ω_j is a highest weight vector in $V_m \otimes V_n$ of the same weight, and we have

$$I(V_m, a; V_n, b)\,(\Omega_j) = \Omega_j' .$$

Further, it is easy to see that, for $j > 0, (x^+ \otimes 1).\Omega_j$ is an $\mathfrak{sl}_2$-highest weight vector of weight $m + n - 2j + 2$; it is non-zero, since otherwise Ω_j would be annihilated by $x^+ \otimes 1$ and by $1 \otimes x^+$, contracting the assumption $j > 0$. Hence, we may assume that

$$(x^+ \otimes 1).\Omega_j = \Omega_{j-1}$$

for $j > 0$. Switching the order of the factors, we have

$$(x^+ \otimes 1).\Omega_j' = -\Omega_{j-1}' .$$

By Proposition 4.2 (and its proof), Ω_j is a Y-highest weight vector in $V_n(b) \otimes V_m(a)$ if

$$b - a = \frac{1}{2}(m+n) - j + 1 .$$

It follows from the formula for the co-multiplication in Definition 1.1 that, in the representation $V_n(b) \otimes V_m(a)$,

$$J(x^+).\Omega_j = \left(b - a - \frac{1}{2}(m + n) + j - 1\right)(x^+ \otimes 1).\Omega_j ,$$

and that in the representation $V_m(a) \otimes V_n(b)$,

$$J(x^+).\Omega_j' = \left(a - b - \frac{1}{2}(m + n) + j - 1\right)(x^+ \otimes 1).\Omega_j' .$$

The equation

$$I(V_m, a; V_n, b)\,(J(x^+).\Omega_j) = J(x^+).(I(V_m, a; V_n, b)\Omega_j)$$

now gives

$$\frac{c_j}{c_{j-1}} = \frac{a - b + \frac{1}{2}(m+n) - j + 1}{a - b - \frac{1}{2}(m+n) - j + 1} .$$

It follows that

(5.10) $$I(V_m, a; V_n, b) = \sum_{j-0}^{\min\{m,n\}} \prod_{i=0}^{j=1} \frac{a - b + \frac{1}{2}(m+n) - i}{a - b - \frac{1}{2}(m+n) + i} P_j .$$

We summarize our results in the following theorem.

THEOREM 5.11. *The R-matrix associated to the representation*

$$V = V_{m_1}(a_1) \otimes \cdots \otimes V_{m_k}(a_k)$$

of Y *is given by*

$$R(a-b) = \Big(\prod_{i,j=1}^{k} I(V_{m_i}, a + a_i; V_{m_j}, b + a_j)\Big)\sigma ,$$

where the intertwining operators are given by equation (5.10) and σ *is the switch map. The order of the factors in the product is such that the* (i, j)*-term appears to the left of the* (i', j')*-term iff*

$$i > i' \quad \text{or} \quad i = i' \quad \text{and} \quad j < j' .$$

6. CONCLUDING REMARKS

Since we have discussed only the Yangian associated to $\mathfrak{sl}_2$ in this paper, it may be worth-while to indicate the extent to which the results above can be generalized to the Yangian $Y(\mathfrak{a})$ associated to an arbitrary finite-dimensional complex simple Lie algebra $\mathfrak{a}$.

The definition of $Y(\mathfrak{a})$ is precisely as in (1.1), except of course that $\{I_\lambda\}$ should be an orthonormal basis of $\mathfrak{a}$ with respect to some invariant inner product. The formulae

$$\tau_a(x) = x , \qquad \tau_a(J(x)) = J(x) + ax ,$$

for $x \in \mathfrak{a}$, again define a one-parameter group of Hopf algebra automorphisms of $Y(\mathfrak{a})$, and the relation, discussed in section 5, between solutions of the quantum Yang-Baxter equation and intertwining operators between tensor products of representations of $Y(\mathfrak{a})$, which follows from the existence of the τ_a, is also valid in the general case.

The major complication which arises in the general case is that, except when $\mathfrak{a} = \mathfrak{sl}_2$, the assignment.

$$x \to x\,, \qquad J(x) \to 0$$

does not extend to a homomorphism of algebras

$$Y(\mathfrak{a}) \to U(\mathfrak{a})\,.$$

However, if $\mathfrak{a} = \mathfrak{sl}_n$, there is a substitute, namely

$$\varepsilon(x) = x\,, \qquad \varepsilon(J(x)) = \frac{1}{4} \sum_{\lambda,\mu} \text{trace}\,(x(I_\lambda I_\mu + I_\mu I_\lambda)) I_\lambda I_\mu$$

(see [4]). One can now define evaluation representations of Y by pulling back representations of $\mathfrak{sl}_n$ using the homomorphisms $\varepsilon \circ \tau_a$. We make the following conjecture, generalizing the case $n = 2$ proved above:

CONJECTURE. *Every finite-dimensional irreducible representation of $Y(\mathfrak{sl}_n)$ is isomorphic to a tensor product of evaluation representations.*

If $\mathfrak{a} \neq \mathfrak{sl}_n$ for any n, it turns out that there is no homomorphism of algebras $\varepsilon: Y(\mathfrak{a}) \to U(\mathfrak{a})$ such that $\varepsilon(x) = x$ for all $x \in U(\mathfrak{a})$ (see [4]), and there is no obvious analogue of the evaluation representations. However, there is a straightforward generalization of Theorem 2.4 classifying the finite-dimensional irreducible representations of $Y(\mathfrak{a})$ in terms of highest weights. Such representations are now in one-to-one correspondence with l-tuples of monic polynomials $\{P_i\}$, where $l = rank\, \mathfrak{a}$. It is natural to define a fundamental representation $V_{i,a}$ of $Y(\mathfrak{a})$ to be one corresponding to a set of polynomials of the form

$$P_j(u) = 1 \quad if \quad j \neq i\,,$$
$$P_i(u) = u - a$$

for some $a \in \mathbf{C}$. As in the case of semisimple Lie algebras, it can be shown that every finite-dimensional irreducible representation of $Y(\mathfrak{a})$ (for any $\mathfrak{a}$) is a subquotient of a tensor product of fundamental representations (see [3]), and understanding the structure of the fundamental representations $V_{i,a}$ is thus an important first step towards understanding the most general representation. The case where the i^{th} node of the Dynkin diagram is extremal (i.e. joined to only one other node), and also the case where the i^{th} fundamental representation of $\mathfrak{a}$ is the adjoint representation, were discussed in [4], and the general case is dealt with in [3], except for a few nodes of E_7 and E_8 (some cases have also appeared without proof in the physics literature [10]).

As we have already mentioned, to any finite-dimensional irreducible representation of any Yangian $Y(\mathfrak{a})$ is associated a rational solution of the QYBE. Many examples of such R-matrices have been computed in the literature (see [10] for a recent summary), although these calculations are mainly restricted to the case of those fundamental representations which are irreducible as representations of $\mathfrak{a}$ (the only exception seems to be [11], which considers the case where i corresponds to the spin representation(s) of $\mathfrak{a} = \mathfrak{so}_n$). In [3] we carry out the computations for most of the remaining fundamental representations, and also for the adjoint representation of $Y(\mathfrak{sl}_n)$. However, the computation of the R-matrix associated to an arbitrary representation of $Y(\mathfrak{a})$ remains a difficult open problem.

REFERENCES

[1] CHARI, V. and A. N. PRESSLEY. New unitary representations of loop groups. *Math. Ann. 275* (1986), 87-104.

[2] CHARI, V. and A. N. PRESSLEY. A new family of irreducible, integrable representations of loop groups, *Math. Ann. 277* (1987), 543-562.

[3] CHARI, V. and A. N. PRESSLEY. Fundamental representations of Yangians and rational R-matrices. To appear in *Journal für reine und angewandte Mathematik*.

[4] DRINFEL'D, V. G. Hopf algebras and the quantum Yang-Baxter equation. *Dokl. Akad Nauk. SSSR 283* (1985) (Russian) = *Soviet Math. Dokl. 32* (1985), 254-258.

[5] —— A new realization of Yangians and quantized affine algebras. *Dokl. Akad. Nauk. SSSR 296* (1987) (Russian) = *Soviet Math. Dokl. 36* (1988), 212-216.

[6] FADDEEV, L. D. Quantum completely integrable models in field theory. Soviet Scientific *Reviews, Section C 1* (1980), 107-155, Harwood Academic Publishers, Chur/New York.

[7] KIRILLOV, A. N. and N. YU RESHETIKHIN. Yangians and Bethe ansatz. *Lett. Math. Phys. 12* (1986), 199-208.

[8] KOREPIN, V. E. The analysis of the bilinear relation of the six-vertex model. *Dokl. Akad. Nauk. SSSR 265* (1982), 1361-1364 (Russian) = *Soviet Phys. Dokl. 27* (1982), 612-613.

[9] KULISH, P. P. YU RESHETIKHIN and E. K. SKLYANIN. Yang-Baxter equation and representation theory I. *Lett. Math. Phys. 5* (1981), 393-403.

[10] OGIEVETSKY, E. and P. WIEGMANN. Factorized S-matrix and the Bethe ansatz for simple Lie groups. *Phys. Lett. 168B* (1986), 360-36.

[11] OGIEVETSKY, N. YU. RESHETIKHIN and P. WIEGMANN. The principal chiral field in two dimensions on classical Lie algebras: The Bethe-ansatz solution and factorized theory of scattering. *Nucl. Phys. B280* (1987), 45-96.

[12] TARASOV, V. O. On the structure of quantum L-operators for the R-matrix of the XXZ-model, *Theoret. Mat. Fiz. 61* (1984), 163-173 (Russian) = *Theoret. Math. Phys.* 61 (1984).

[13] —— Irreducible monodromy matrices for the R-matrix of the XXZ-model and lattice local quantum Hamiltonians. *Theoret. Mat. Fiz. 63* (1985), 175-196 (Russian) = *Theoret. Math. Phys.* 63 (1985).

(Reçu le 9 novembre 1989)

Vyjayanthi Chari

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay 400 005, India

Andrew Pressley

Department of Mathematics
King's College, Strand
London WC2R 2LS
England U.K.