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by Tarasov [12], [13], for the case of $Y(\mathfrak{sl}_2)$, using ideas of Korepin [8], and was extended by Drinfel'd [5] to the general case. The evaluation representations and their tensor products appeared implicitly in the work of Kulish, Reshetikhin and Sklyanin [9] mentioned above, but they did not prove that all finite-dimensional irreducible representations are of this form. Our determination of the precise conditions under which such tensor products are irreducible is also new.

One of the difficulties of this subject is the unfamiliarity of the language in which many of the fundamental papers are written, which is that of quantum inverse scattering theory and exactly solvable models in statistical mechanics. We have tried in our presentation to express the results in more conventional mathematical language. In fact, all that is required is some familiarity with the basic techniques of Lie theory.

1. Yangians

We begin with the definition of the Yangian taken from [4]. Let $\{I_\lambda\}$ be an orthonormal basis of $\mathfrak{sl}_2$ with respect to some invariant inner product (,); for example, using the trace form

$$(x, y) = \operatorname{trace}(xy) ,$$

one can take the basis $\left\{\frac{x^+ + x^-}{\sqrt{2}}, \frac{i(x^+ - x^-)}{\sqrt{2}}, \frac{h}{\sqrt{2}}\right\}$, where $\{x^+, x^-, h\}$ is the usual basis:

$$[h, x^\pm] = \pm 2x^\pm , \quad [x^+, x^-] = h .$$

Definition 1.1. The *Yangian* $Y = Y(\mathfrak{sl}_2)$ associated to $\mathfrak{sl}_2$ is the Hopf algebra over $\mathbf{C}$ generated (as an associative algebra) by $\mathfrak{sl}_2$ and elements $J(x)$ for $x \in \mathfrak{sl}_2$ with relations

(1) $[x, J(y)] = J([x, y]), \quad J(ax + by) = aJ(x) + bJ(y), \; a, b \in \mathbf{C}$,

(2) $$[J(x), J([y, z])] + \text{cyclic permutations of } x, y, z$$
$$= ([x, I_\lambda], [[y, I_\mu], [z, I_\nu]]) \{I_\lambda, I_\mu, I_\nu\} ,$$

(3) $$[[J(x), J(y)], [z, J(w)]] + [[J(z), J(w)], [x, J(y)]]$$
$$= (([x, I_\lambda], [[y, I_\mu], [[z, w], I_\nu]]) + ([z, I_\lambda], [[w, I_\mu], [[x, y], I_\nu]])) \{I_\lambda, I_\mu, I_\nu\},$$

where repeated indices are summed over and

$$\{x_1, x_2, x_3\} = \sum_\pi x_{\pi(1)} x_{\pi(2)} x_{\pi(3)} ,$$

the sum being over all permutations π of $\{1, 2, 3\}$. The co-multiplication map of Y is given by

$$\text{(4)} \qquad \Delta(x) = x \otimes 1 + 1 \otimes x ,$$
$$\Delta(J(x)) = J(x) \otimes 1 + 1 \otimes J(x) + \frac{1}{2}[x \otimes 1, \Omega] ,$$

where Ω is as defined in equation (0.7).

Remarks.

1. Relation (2) is actually a consequence of the other relations. We have kept it because relations (1), (2) and (3) are the defining relations of the Yangian associated to an arbitrary finite-dimensional Lie algebra $\mathfrak{g}$, and in the general case (2) is not redundant.

2. The relations depend on the choice of inner product (,) but, up to isomorphism, the Hopf algebra Y does not.

There is another realization of Y due to Drinfel'd [5, Theorem 1] which we shall need in the discussion of highest weight representations of Y in the next section.

THEOREM 1.2. *Y is isomorphic to the associative algebra over $\mathbf{C}$ with generators x_k^+, x_k^-, h_k for $k = 0, 1, \ldots$ and relations*

(1) $[h_k, h_l] = 0, \quad [h_0, x_k^\pm] = \pm 2x_k^\pm, \quad [x_k^+, x_l^-] = h_{k+l}$;

(2) $[h_{k+1}, x_l^\pm] - [h_k, x_{l+1}^\pm] = \pm (h_k x_l^\pm + x_l^\pm h_k)$;

(3) $[x_{k+1}^\pm, x_l^\pm] - [x_k^\pm, x_{l+1}^\pm] = \pm (x_k^\pm x_l^\pm + x_l^\pm x_k^\pm)$.

The isomorphism ϕ between the two realizations of Y is given by

$$\phi(h) = h_0, \quad \phi(x^\pm) = x_0^\pm ,$$
$$\phi(J(h)) = h_1 + \frac{1}{2}(x_0^+ x_0^- + x_0^- x_0^+ - h_0^2) ,$$
$$\phi(J(x^\pm)) = x_1^\pm - \frac{1}{4}(x_0^\pm h + h x_0^\pm) .$$

One of the difficulties which arises in using this realization of Y is that no explicit formula for the co-multiplication map Δ on the generators $h_k, x_k^\pm$ is known. However, the following formulas follow easily from the formulae in Definition 1.1 and Proposition 1.2:

$$\Delta(h_0) = h_0 \otimes 1 + 1 \otimes h_0 ,$$
$$\Delta(h_1) = h_1 \otimes 1 + h_0 \otimes h_0 + 1 \otimes h_1 - 2x_0^- \otimes x_0^+$$
$$\Delta(x_0^+) = x_0^+ \otimes 1 + 1 \otimes x_0^+ ,$$

$$(1.3)\qquad \begin{aligned}\Delta(x_1^+) &= x_1^+ \otimes 1 + 1 \otimes x_1^+ + h_0 \otimes x_0^+\\ \Delta(x_0^-) &= x_0^- \otimes 1 + 1 \otimes x_0^- ,\\ \Delta(x_1^-) &= x_1^- \otimes 1 + 1 \otimes x_1^- + x_0^- \otimes h_0 .\end{aligned}$$

As an application of these formulas we shall prove the following useful result.

PROPOSITION 1.4. *The assignment* $x_k^+ \mapsto x_k^-, x_k^- \mapsto x_k^+, h_k \mapsto h_k, k \in \mathbf{Z}_+$, *extends to an anti-homomorphism* $\omega: Y \to Y$. *Moreover, the following diagram is commutative:*

$$\begin{array}{ccc} Y & \xrightarrow{\Delta'} & Y \otimes Y \\ \omega \downarrow & & \downarrow \omega\otimes\omega \\ Y & \xrightarrow[\Delta]{} & Y \otimes Y \end{array}$$

Proof. The fact that ω extends to an anti-homomorphism of Y follows almost immediately from the relations in Theorem 1.2. To prove that the diagram is commutative, it is enough to check that $\Delta\omega$ and $(\omega \otimes \omega)\Delta'$ agree on a set of generators of Y. From the relations in (1.2) and the form of the isomorphism ϕ, it is clear that Y is generated by $h_0, x_0^\pm$ and $x_1^\pm$. For $h_0, x_0^\pm$ the verification is trivial. From equations (1.3) we have

$$\begin{aligned}\Delta\omega(x_1^-) &= \Delta(x_1^+)\\ &= x_1^+ \otimes 1 + 1 \otimes x_1^+ + h_0 \otimes x_0^+ .\end{aligned}$$

On the other hand,

$$\begin{aligned}(\omega \otimes \omega)\Delta'(x_1^-) &= (\omega \otimes \omega)\,(x_1^- \otimes 1 + 1 \otimes x_1^- + h_0 \otimes x_0^-)\\ &= x_1^+ \otimes 1 + 1 \otimes x_1^+ + h_0 \otimes x_0^+ .\end{aligned}$$

The proof for x_1^+ is similar.

Definition 1.5. Let H (resp. $N^\pm$) denote the subalgebra of Y generated by the h_k (resp. $x_k^\pm$) for $k \in \mathbf{Z}_+$.

We shall now give a more precise description of the co-multiplication map.

PROPOSITION 1.6. *The co-multiplication map* Δ *of* Y *satisfies:*

(1) $\Delta(h_k) = h_k \otimes 1 + h_{k-1} \otimes h_1 + h_{k-2} \otimes h_1 + \cdots + h_0 \otimes h_{k-1} + 1 \otimes h_k$ *modulo* $\sum_{p\geq 0} Y \otimes Yx_p^+ + Yx_p^+ \otimes Y$;

(2) $$\Delta(x_k^+) = x_k^+ \otimes 1 + h_0 \otimes x_{k-1}^+ + h_1 \otimes x_{k-2}^+ + \cdots + h_{k-1} \otimes x_0^+ + 1 \otimes x_k^+ \ \textit{modulo} \ \textstyle\sum_{p,q,r\geqslant 0} Yx_p^- \otimes Yx_q^+ x_r^+ ;$$

(3) $$\Delta(x_k^-) = x_k^- \otimes 1 + x_{k-1}^- \otimes h_0 + x_{k-2}^- \otimes h_1 + \cdots + x_0^- \otimes h_{k-1} + 1 \otimes x_k^- \ \textit{modulo} \ \textstyle\sum_{p,q,r\geqslant 0} Yx_p^- x_q^- \otimes x_r^+ .$$

For the proof, we shall need

LEMMA 1.7. *For all* $k, l \in \mathbf{Z}_+$, *we have* $x_k^+ h_l \in HN^+$ *and* $h_l x_k^- \in N^- H$.

Proof. We prove the first formula by induction on l; the second follows from the first by Proposition 1.4. If $l = 0$, then by (1.2) (1),

$$x_k^+ h_0 = h_0 x_k^+ - 2x_k^+$$

which is in HN^+. Next, by (1.2) (2),

$$\begin{aligned}[h_{l+1}, x_k^+] &= [h_l, x_{k+1}^+] + h_l x_k^+ + x_k^+ h_l \\ &= h_l(x_{k+1}^+ + x_k^+) + (x_k^+ - x_{k+1}^+)h_l .\end{aligned}$$

Hence,

$$x_k^+ h_{l+1} = h_{l+1} x_k^+ - h_l(x_{k+1}^+ + x_k^+) + (x_{k+1}^+ - x_k^+) h_l ,$$

which belongs to HN^+ by the induction hypothesis.

Proof of Proposition 1.6. It is enough to prove formula (2). For (3) follows from (2) by Proposition 1.4. Also,

$$\begin{aligned}\Delta(h_k) &= \Delta([x_k^+, x_0^-]) \\ &= [\Delta(x_k^+), x_0^- \otimes 1 + 1 \otimes x_0^-] \\ &= h_k \otimes 1 + h_{k-1} \otimes h_0 + \cdots + 1 \otimes h_k - 2\sum_{i=0}^{k-1} x_i^- \otimes x_{k-i+1}^+ \\ &\qquad \textit{modulo} \ \textstyle\sum_{p,q,r\geqslant 0} [Yx_p^- \otimes Yx_p^+ x_r^+, x^- \otimes 1 + 1 \otimes x^-] .\end{aligned}$$

To prove (1), it therefore suffices to prove that $x_q^+ x_r^+ x_0^- \in \sum_{s\geqslant 0} Yx_s^+$. Since

$$x_q^+ x_r^+ x_0^- = x_q^+ h_r + x_q^+ x_0^- x_r^+ ,$$

this follows from Lemma (1.7).

To prove (2), define $\tilde{h}_1 = h_1 - \frac{1}{2} h_0^2$. Then:

(1.8) $$\Delta(\tilde{h}_1) = \tilde{h}_1 \otimes 1 + 1 \otimes \tilde{h}_1 - 2x_0^- \otimes x_0^+ ,$$

(1.9) $$[\tilde{h}_1, x_k^+] = 2x_{k+1}^+ ,$$

(1.10) $$[\tilde{h}_1, x_k^-] = -2x_{k+1}^- .$$

In fact, (1.8) follows from (1.3) and (1.9) is proved by induction on k, using the relation

$$[h_1, x_k^+] - [h_0, x_{k+1}^+] = h_0 x_k^+ + x_k^+ h_0 ,$$

the right-hand side of which is $\left[\frac{1}{2} h_0^2, x_k^+\right]$. Finally, (1.10) follows from (1.9) and (1.4).

The proof of (2) now proceeds by induction on k. The result is known for $k = 0$ and 1. For the inductive step, we use (1.8) to obtain

$$\begin{aligned} 2\Delta(x_{k+1}^+) &= \Delta([\tilde{h}_1, x_k^+]) \\ &= [\tilde{h}_1 \otimes 1 + 1 \otimes \tilde{h}_1 - 2x_0^- \otimes x_0^+ , x_k^+ \otimes 1 + 1 \otimes x_k^+ \\ &\quad + \sum_{i=0}^{k-1} h_i \otimes x_{k-i-1}^+ + R] , \end{aligned}$$

where the remainder term $R \in \sum_{p,q,r \geqslant 0} Yx_p^- \otimes Yx_q^+ x_r^+$. Hence, using (1.9),

$$\Delta(x_{k+1}^+) = x_{k+1}^+ \otimes 1 + 1 \otimes x_{k+1}^+ + \sum_{i=0}^{k} h_i \otimes x_{k-i}^+ + R'$$

where

$$\begin{aligned} R' &= \frac{1}{2} [\tilde{h}_1 \otimes 1 + 1 \otimes \tilde{h}_1 - 2x_0^- \otimes x_0^+ , R] - x_0^- \otimes [x_0^+ , x_k^+] \\ &\quad - \sum_{i=0}^{k-1} (h_i x_0^- \otimes [x_0^+ , x_{k-i-1}^+] + 2x_i^- \otimes x_0^+ x_{k-i-1}^+) . \end{aligned}$$

It suffices to check that the first term belongs to $\sum_{p,q,r \geqslant 0} Yx_p^- \otimes Yx_q^+ x_r^+$, and this follows easily from (1.9) and (1.10). This completes the proof.

Finally, we shall need the following analogue of the easy half of the Poincaré-Birkhoff-Witt theorem.

PROPOSITION 1.11. $Y = N^- . H . N^+$.

Proof. The proof is the same as for Lie algebras. Choose any total ordering $\prec$ on the generating set $\{x_k^\pm , h_k\}_{k \in \mathbf{Z}_+}$ such that $x_k^- \prec h_l \prec x_m^+$ for all $k, l, m \in \mathbf{Z}_+$. If $u = u_1 u_2 \dots u_n$ is any monomial in the generators of degree n, define its index

$$\operatorname{ind}(u) = \sum_{i<j} \varepsilon_{ij}$$

where

$$\varepsilon_{ij} = \begin{cases} 0 & \text{if} \quad u_i \prec u_j \\ 1 & \text{if} \quad u_j \prec u_i . \end{cases}$$

Using Lemma 1.7, each monomial can be written as a sum of monomials of smaller degree, or smaller index, and hence, by an obvious induction, as a sum of monomials of index zero.

2. Highest weight representations

By analogy with the definition of highest weight representations of semi-simple Lie algebras, one makes the following

Definition 2.1. A representation V of the Yangian Y is said to be *highest weight* if there is a vector $\Omega \in V$ such that $V = Y\Omega$ and

$$x_k^+ \Omega = 0, \qquad h_k \Omega = d_k \Omega, \qquad k = 0, 1, \dots$$

for some sequence of complex numbers $\mathbf{d} = (d_0, d_1, \dots)$. In this case, Ω is called a highest weight vector of V and $\mathbf{d}$ its highest weight.

Remark. It follows immediately from Definition 1.1 that the assignment $x \mapsto x$ for $x \in \mathfrak{sl}_2$ extends to a homomorphism of algebras $\iota: U(\mathfrak{sl}_2) \to Y$. By Proposition 2.5 below, ι is injective. Thus, any representation of Y can be restricted to give a representation of $\mathfrak{sl}_2$. In particular, we can speak of weights relative to $\mathfrak{sl}_2$ as well as relative to Y. It will always be clear from the context which type of weight is intended.

As in the case of semi-simple Lie algebras, there is a universal highest weight representation of Y of any given highest weight:

Definition 2.2. Let $\mathbf{d} = (d_0, d_1, \dots)$ be any sequence of complex numbers. The *Verma representation* $M(\mathbf{d})$ is the quotient of Y by the left ideal generated by $\{x_k^+, h_k - d_k \cdot 1\}_{k \in \mathbf{Z}_+}$.

PROPOSITION 2.3. *The Verma representation* $M(\mathbf{d})$ *is a highest weight representation with highest weight* $\mathbf{d}$, *and every such representation is*