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PRIME TO A FIXED INTEGER  $k$   
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$$H(k, x) := \sum_{n \geq 1} \frac{\gamma_k(n)}{n} \left( \frac{1}{2} - \left\{ \frac{x}{n} \right\} \right)$$

then

$$(0.14) \quad H(k, x) = H(k, [x]) - \frac{k}{\sigma(k)} \{x\} + o(1),$$

and

LEMMA 0. *We have*

$$(0.15) \quad S_k = \sup_{n \in \mathbf{Z}} H(k, n).$$

*Proof.* In view of (0.13), (0.14), and the definition of  $H(k, x)$ , it is sufficient to show that

$$(0.16) \quad \limsup_{N \rightarrow \infty, N \in \mathbf{N}} H(k, N) = \sup_{n \in \mathbf{Z}} H(k, n).$$

When  $k = 1$  this is easily verified; when  $k \geq 2$  and  $N \in \mathbf{Z}$  we define for each positive integer  $i$  the positive integer  $N_i := (|N| + 1)k^i + N$  and we see, since

$$(0.17) \quad \sum_{m \nmid k^i} \frac{\gamma_k(m)}{m} \rightarrow 0 \quad (i \rightarrow \infty),$$

and since for every divisor  $m$  of  $k^i$  we have  $\{N_i/m\} = \{N/m\}$ , that

$$(0.18) \quad \lim_{i \rightarrow \infty} H(k, N_i) = H(k, N). \quad \square$$

## 1. PROOF OF THEOREM 1

We first set some terminology. Let  $g: [1, \infty] \rightarrow \mathbf{R}$  be a measurable function, and consider as in [P1]

$$(1.1) \quad D_0(u) = D_{0,g}(u) := \lim_{x \rightarrow \infty} \frac{1}{x} \mu \{t \in [0, x], g(t) \leq x\},$$

and

$$(1.2) \quad D_0(u^+) := \lim_{\substack{v \rightarrow u \\ v \in E}} D_0(v), \quad D_0(u^-) := \lim_{\substack{v \rightarrow u \\ v \in E}} D_0(v),$$

where  $\mu$  denotes the Lebesgue measure and  $E$  the set of values for which  $D_0$  exists. In case  $D_0$  exists almost everywhere we say, following A. Wintner [W, p. 537], that  $g$  possesses an asymptotic distribution function. If (and only if) this is so we define an associated function  $D = D_g: \mathbf{R} \rightarrow [0, 1]$  by

$$(1.3) \quad D(u) := \frac{1}{2} (D_0(u^+) + D_0(u^-)) .$$

And it is this function  $D$  we call *the asymptotic distribution function of  $g$* . The convention is of course abusive<sup>2</sup>); we point out however that  $D_0$  exists and coincides with  $D$  at least wherever  $D$  is continuous (which, since  $D$  is a distribution function, is the case almost everywhere).

The first two statements of Theorem 1,  $D = D_h$  exists and is continuous, are proved through a straightforward application of two theorems from [P1].

Indeed, it is easy to see that

$$(1.4) \quad \sum_{n \leq x} \gamma_k(n) = O((\log x)^{\omega(k)}) = o(x) \quad (x \rightarrow \infty)$$

holds, and that for any function  $z = z(x) \rightarrow \infty$  ( $x \rightarrow \infty$ ) (and in particular for a slowly increasing function), we have

$$(1.5) \quad H(k, x) = \sum_{n \leq z} \frac{\gamma_k(n)}{n} \left( -\psi \left( \frac{x}{n} \right) \right) + o(1) ,$$

where  $\psi(y)$  denotes the function  $\{y\} - \frac{1}{2}$  which satisfies

$$(1.6) \quad \int_0^1 \psi(t) dt = 0 .$$

In the notation of [P1] the properties (1.4) through (1.6) are expressed by writing  $h \in C_z(\psi_k, -\psi)$ . Thus from Theorem 4 of that paper we have the existence of  $D_h$ . And since  $\psi$  is odd almost everywhere Theorem 5 of [P1] tells us that  $D_h$  is symmetric.

We pass now to the third assertion of the theorem, namely that  $I_k = -S_k$ . We denote by  $S$  the bounded support of  $D_h$  and by  $-s$  and  $s$  its

<sup>2</sup>) Its purpose is to ensure that  $D$  be *normalized*, i.e. that the relation

$$D(u) = \frac{1}{2} (D(u^+) + D(u^-))$$

hold for every real number  $u$ .

greatest lower bound and least upper bound: we have  $I_k \leq -s < s \leq S_k$ .

We show that

$$(1.7) \quad I_k = -s = -S_k$$

holds by ensuring that

$$(1.8) \quad 0 < D_h(\alpha) < 1 \quad \text{for every } \alpha \in (I_k, S_k).$$

We prove here that  $D_h(S_k - \varepsilon) < 1$  for every  $\varepsilon > 0$ ; the rest of the proof is similar. There is an increasing sequence of natural numbers  $n_i$  with  $H(k, n_i) \rightarrow S_k$  ( $i \rightarrow \infty$ ), and thus we may select some natural number  $N$  satisfying

$$(1.9) \quad H(k, N) > S_k - \frac{\varepsilon}{4}$$

and

$$(1.10) \quad \frac{1}{2} \sum_{n > N} \frac{|\gamma_k(n)|}{n} < \frac{\varepsilon}{4}.$$

Hence if we define

$$(1.11) \quad H^*(k, N, M) := \sum_{n \leq N} \frac{\gamma_k(n)}{n} \left( \frac{1}{2} - \left\{ \frac{M}{n} \right\} \right).$$

we have

$$(1.12) \quad H^*(k, N, N) > S_k - \frac{\varepsilon}{2}.$$

Also, if  $L$  is the least common multiple of the integers  $1, 2, \dots, N$ , then

$$(1.13) \quad H^*(k, N, mL + N) = H^*(k, N, N)$$

for every integer  $m$ , and it follows from (1.12) and (1.10) that

$$(1.14) \quad H(k, mL + N) > S_k - \frac{3\varepsilon}{4}$$

for every integer  $m$ . Now since  $D_{0,h}$  exists and coincides with  $D_h$  almost everywhere we can find two numbers  $\beta$  and  $\gamma$  satisfying

$$(1.15) \quad S_k - \varepsilon \leq \beta < \beta + \frac{\varepsilon}{5} \leq \gamma \leq S_k - \frac{3\varepsilon}{4}$$

and

$$(1.16) \quad D_h(\delta) = D_{0,h}(\delta) \quad (\delta = \beta \text{ or } \gamma).$$

In view of (0.14) this implies that

$$(1.17) \quad \begin{aligned} 1 - D_h(S_k - \varepsilon) &\geq D_h\left(S_k - \frac{3\varepsilon}{4}\right) - D_h(S_k - \varepsilon) \\ &\geq D_h(\gamma) - D_h(\beta) = D_{0,h}(\gamma) - D_{0,h}(\beta) \geq \frac{1}{L} \cdot \frac{\varepsilon}{5} \cdot \frac{\sigma(k)}{k}. \quad \square \end{aligned}$$

*Remark.* I studied in [P2] an error term associated with the  $k$ -th Jordan totient function (for  $k \geq 2$ ), that can be expressed in terms of the function

$$(1.18) \quad g_k(x) := - \sum_{n=1}^{\infty} \frac{\mu(n)}{n^k} \psi\left(\frac{x}{n}\right),$$

where  $\mu$  denotes the Moebius function, and I proved by a direct method that

$$(1.19) \quad \liminf_{x \rightarrow \infty} g_k(x) = - \limsup_{x \rightarrow \infty} g_k(x).$$

This can also be obtained by an argument similar to the above proof.

## 2. THE CASE $\omega(k) = 2$

In this section we obtain an estimate more general than (0.10) of Theorem 2.

**THEOREM 2'.** *Let  $k = pq$  where  $p < q$  and  $p$  and  $q$  are prime numbers, and let  $d = q - ps$  with  $1 \leq d \leq p - 1$  be the remainder of the Euclidean division of  $q$  by  $p$ . Then we have*

$$(2.1) \quad S_k \geq \frac{k}{\sigma(k)} + \frac{1}{(p+1)} - \frac{pd}{(p+1)(q+1)} + \frac{(p+1)(p-2)(q-1)}{p^2q}.$$

The right side of (2.1) is easily seen to exceed  $k/\sigma(k)$  for any  $p$  and  $q$ . And in the special case where  $p = 2$  it reduces to  $\left(q - \frac{1}{3}\right)/(q+1)$ .

*Proof.* Let  $N$  be a positive integer. We define, modulo  $p^N q^N$ , the integer  $x = x_N$  by the system of congruences

$$(2.2) \quad \begin{cases} x \equiv -1(p^N) \\ x \equiv -d-1(q^N). \end{cases}$$