Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 37 (1991)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: DESCARTES' THEOREM IN n DIMENSIONS
Autor: Grünbaum, Branko / Shephard, G. C.
Bibliographie
DOI: https://doi.org/10.5169/seals-58726

$$\sum_i \delta(V_i) = f_0(P) - s_0(P)$$
$$\sum_j \delta(E_j) = f_1(P) - s_1(P) ,$$

so (3) becomes

$$(f_0(P) - \sum_i \delta(V_i)) - (f_1(P) - \sum_j \delta(E_j)) + f_2(P) = f_3(P)$$

or,

$$\sum_i \delta(V_i) - \sum_j \delta(E_j) = f_0(P) - f_1(P) + f_2(P) - f_3(P) = \chi(P)$$

which is (1), and the theorem is proved.

An exactly analogous argument holds in n dimensions (for all $n \geqslant 3$). In the proof we use the $(n-1)$-dimensional form of Gram's Theorem (see [3], [7]) for each $(n-1)$-face F_k of P:

$$s_0(F_k) - s_1(F_k) + s_2(F_k) - \dots + (-1)^{n-2} s_{n-2}(F_k) = (-1)^n$$

with a notation analogous to that used in (2). This leads to the statement:

DESCARTES' THEOREM IN n DIMENSIONS. *Let P be an elementary polytope in E^n, let $\delta(F_i^m) = 1 - s(F_i^m)$ be the deficiency of P at the m-face F_i^m of P $(m = 0, 1, \dots, n-3)$, and let $\delta_m(P) = \sum_i \delta(F_i^m)$, where summation is over all the m-faces F_i^m of P. Then*

$$\sum_{m=0}^{n-3} (-1)^m \delta_m(P) = \chi(P) ,$$

where $\chi(P)$ is the Euler characteristic of P.

REFERENCES

[1] BARNETTE, D. W., P. GRITZMANN and R. HÖHNE. On valences of polyhedra. (To appear.)

[2] FEDERICO, P. J. Descartes on polyhedra, a study of the *De Solidorum Elementis* (Volume 4 in *Sources in the History of Mathematics and Physical Sciences*), Springer-Verlag, New York-Heidelberg-Berlin 1982.

[3] GRÜNBAUM, B. *Convex Polytopes*. Wiley, 1967.

[4] GRÜNBAUM, B. and G. C. SHEPHARD. A dual for Descartes' theorem on polyhedra. *Math. Gazette 71* (1987), 214-216.

[5] HILTON, P. and J. PEDERSEN. Descartes, Euler, Poincaré, Pólya – and polyhedra. *L'Enseignement Math. 27* (1981), 327-343.

[6] HILTON, P. and J. PEDERSEN. Duality and Descartes' deficiency. *Computers Math. Applic. 17* (1989), 73-88.
[7] SHEPHARD, G. C. An elementary proof of Gram's theorem for convex polytopes. *Canad. J. Math. 19* (1967), 1214-1217.
[8] —— Angle deficiencies for convex polytopes. *J. London Math. Soc. 43* (1968), 325-336.

(Reçu le 5 juin 1990)

Branko Grünbaum

University of Washington GN-50
Seattle, WA 98195, USA

G. C. Shephard

University of East Anglia
Norwich NR4 7TJ, England

vide-leer-empty