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Artikel: ON THE DIFFEOMORPHISM GROUPS OF CERTAIN ALGEBRAIC SURFACES

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1. Construction of diffeomorphisms

Let X be a simply connected smooth compact complex algebraic surface with quadratic intersection form $q_X: H_2(X, \mathbf{Z}) \to \mathbf{Z}$ and canonical class $k_X \in H^2(X, \mathbf{Z}) = \operatorname{Hom}(H_2(X, \mathbf{Z}), \mathbf{Z})$. The second homology group $H_2(X, \mathbf{Z})$ endowed with the quadratic form q_X forms a lattice which we denote by L. Let $O(L)$ be the corresponding group of isometries.

We introduce some notation: Let $b_2^+(X)$ denote the dimension of any maximal subspace of $L_\mathbf{R} = L \otimes \mathbf{R}$ on which q_X is positive definite. Note that $b_2^+(X) = 2p_g(X) + 1$ where $p_g(X)$ is the geometric genus of X. The set of all oriented maximal positive definite subspaces of $L_\mathbf{R}$ forms an open subset Ω of the Grassmannian $G^{\text{or}}(b_2^+(X), L_\mathbf{R})$ of oriented $b_2^+(X)$-dimensional subspaces of the vector space $L_\mathbf{R}$. It has two components if q_X is indefinite. We define $O'(L)$ to be the subgroup of $O(L)$ consisting of those automorphisms which leave each component of Ω invariant. (For an equivalent definition of $O'(L)$ see [E2, 4.1].) Let $O_k(L)$ be the subgroup of $O(L)$ consisting of automorphisms preserving $k = k_X$. Finally we define $O'_k(L) := O_k(L) \cap O'(L)$.

An important subset of $\operatorname{Diff}_+(X)$ is the set of classes of diffeomorphisms obtained by monodromy transformations of a smooth family containing X as a fibre. By a *smooth family* we mean a smooth (in the analytic category) proper holomorphic mapping $\pi: \mathscr{X} \to T$ of connected complex spaces $\mathscr{X}$ and T; π is the projection of a locally trivial differentiable fibre bundle, so that for a point $t_0 \in T$ with $X = \pi^{-1}(t_0)$ there is a *monodromy representation* $\rho: \pi_1(T, t_0) \to \operatorname{Diff}_+(X)$. The image Γ of $\psi \circ \rho$ in $O(L)$ is called the *monodromy group* of the smooth family. The monodromy group preserves k_X. It also preserves the components of Ω: to see this consider a loop τ representing an element in $\pi_1(T, t_0)$ and let $g_t: X_{t_0} = X \to X_t$ be the diffeomorphisms corresponding to $t \in \tau$. The mapping $t \mapsto (g_t)_*(\alpha) \in G^{\text{or}}(b_2^+(X), L_\mathbf{R})$ is continuous for every $\alpha \in \Omega$, hence $\Gamma \subset O'_k(L)$.

For certain algebraic surfaces there exist smooth families whose monodromy group is the whole group $O'_k(L)$. This is summarized in the following theorem. Among these surfaces there are certain iterated ramified coverings of $\mathbf{P}^1 \times \mathbf{P}^1$ or $\mathbf{P}^2$ which were studied by Moishezon [M] and Salvetti [S]: Let $(n_1, \dots, n_r)$ be a sequence of positive integers. For Moishezon's construction, let $X_0 = \mathbf{P}^1 \times \mathbf{P}^1$ and let $C \subset \mathbf{P}^1 \times \mathbf{P}^1$ be the divisor $\{\mathrm{pt}\} \times \mathbf{P}^1 + \mathbf{P}^1 \times \{\mathrm{pt}\}$. Construct a sequence $\beta_i : X_i \to X_{i-1}$ of cyclic coverings β_i of degree 3 with nonsingular branch locus linearly equivalent to $(\beta_{i-1} \circ \dots \circ \beta_1)^*(3n_i C)$. Let $X_r = X_r(n_1, \dots, n_r)$. We call a surface $X_r(n_1, \dots, n_r)$ with $n_i \geqslant 2$ for some i, $1 \leqslant i \leqslant r$, (cf. [M, §4, Remark 3]) a *Moishezon surface.* For Salvetti's construction we need in addition to $(n_1, \dots, n_r)$ a sequence $(d_1, \dots, d_r)$ of positive integers satisfying $d_i \mid n_i$ for all $i = 1, \dots, r$. Let $Y_0 = \mathbf{P}^2$, and choose smooth curves $C_i \subset \mathbf{P}^2$ of degree n_i, $i = 1, \dots, r$, so that $C := C_1 \cup \dots \cup C_r$ has only normal crossings. Construct a sequence $\beta_i : Y_i \to Y_{i-1}$ of cyclic coverings β_i of degree d_i ramified over $(\beta_{i-1} \circ \dots \circ \beta_1)^*(C_i)$. A *Salvetti surface* is a surface $Y_r = Y_r(n_1, \dots, n_r; d_1, \dots, d_r)$, if at least one $n_i > 5$ with the corresponding $d_i \geqslant 3$ (cf. [S, Corollary to Proposition 2]).

THEOREM 1. *Let X be a simply connected algebraic surface with $p_g(X) > 0$, and suppose that X is either*

(i) *a complete intersection, or*

(ii) *a Moishezon or Salvetti surface.*

Then there is a smooth family $\pi : \mathscr{X} \to T$ with $\pi^{-1}(t_0) = X$ for some $t_0 \in T$ with monodromy group

$$\Gamma = O_k'(L) .$$

For complete intersection surfaces this is proved in [B, E2, E3], for Moishezon surfaces see [M], and for Salvetti surfaces see [S], together with [EO, Theorem 2.5 and §1].

Next we construct an orientation preserving diffeomorphism $\sigma : X \to X$ with $\sigma^*(k_X) = -k_X$. Suppose X is embedded in a complex projective space $\mathbf{P}^N$ so that X is the zero locus of a finite set $\{f_1, \dots, f_M\}$ of homogeneous polynomials in the coordinates $z_0, \dots, z_N$ of $\mathbf{P}^N$.

Denote by $\sigma : \mathbf{C} \to \mathbf{C}$ complex conjugation, and let $X^\sigma \subset \mathbf{P}^N$ be the zero locus of $\{f_1^\sigma, \dots, f_M^\sigma\}$, where f_i^σ is obtained from f_i by applying σ to all the coefficients. Then $\sigma : \mathbf{P}^N \to \mathbf{P}^N$ induces a diffeomorphism $X \to X^\sigma$, also denoted by σ, which satisfies $\sigma^*(k_{X^\sigma}) = -k_X$. If X is given by real equations $f_1, \dots, f_M$, then $\sigma : X \to X^\sigma = X$ is a self-diffeomorphism of X. Such equations can be found if X is a complete intersection and if X is a

Moishezon or Salvetti surface. (In the latter case the branch locus must be given by real equations.)

Therefore we have:

COROLLARY 2. *Let* X *be an algebraic surface as in Theorem 1. Then*

$$O'_k(L) \cdot \{\sigma_*, \mathrm{id}\} \subset \psi(\mathrm{Diff}_+(X)) .$$

2. INVARIANCE OF THE CANONICAL CLASS

S.K. Donaldson [D] has defined a series of invariants for certain smooth 4-manifolds. They are in particular defined for simply connected algebraic surfaces X with $p_g(X) > 0$. We assume from now on that X is such a surface. There are two types of invariants according to the gauge group being $SU(2)$ or $SO(3)$.

Let us first recall the $SU(2)$-case. Principal $SU(2)$-bundles over X are classified by their second Chern class $c_2(P)$. For each $l > l_0$, using such a bundle with $c_2(P) = l$, Donaldson defines a polynomial

$$\Phi_l(X) : \mathrm{Sym}^d(L) \to \mathbf{Z}$$

of degree $d = d(l) = 4l - 3(p_g(X) + 1)$, which depends only on the underlying C^∞-structure of X and is invariant up to sign under $\psi(\mathrm{Diff}_+(X))$. Donaldson shows that these invariants are nontrivial for all sufficiently large l [D].

We will need the slightly more complicated $SO(3)$-invariants. The simple Lie group $SO(3)$ is isomorphic to $PU(2)$, so that one has an exact sequence

$$1 \to S^1 \to U(2) \to SO(3) \to 1 .$$

Let P be a principal $SO(3)$-bundle over X. Such a bundle has two characteristic classes which determine it up to isomorphism: the second Stiefel-Whitney class $w_2(P) \in H^2(X, \mathbf{Z}/2)$ and the first Pontryagin class $p_1(P) \in H^4(X, \mathbf{Z})$.

Suppose that $w_2(P)$ is nonzero and choose an integral lifting c of $w_2(P)$, i.e. $c \in H^2(X, \mathbf{Z}), \bar{c} = w_2(P)$ (here $\bar{c}$ means the reduction of c modulo 2). Such a lifting exists since X is simply connected, and determines a $U(2)$-lifting $\hat{P}$ of P, i.e. a $U(2)$-bundle $\hat{P}$ with $\hat{P}/S^1 = P$ and with $c = c_1(\hat{P})$ [HH]. The Chern classes of $\hat{P}$ are related to the characteristic classes of P by $w_2(P) = \bar{c}_1(\hat{P})$ and $p_1(P) = c_1^2(\hat{P}) - 4c_2(\hat{P})$. In addition to this choose an element $\alpha \in \Omega$. Donaldson shows that these choices give rise to a polynomial

$$\Phi_{c,\alpha,P}(X) : \mathrm{Sym}^d(L) \to \mathbf{Z}$$