2. The operator T

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In these coordinates we can write down the $q+1$ vertices of X that are adjacent to a typical vertex $\begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix}$. They are

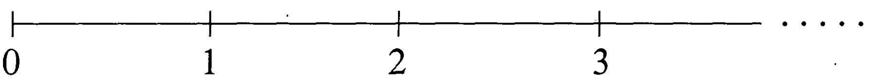
$$\begin{pmatrix} t^{n+1} & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} t^{n-1} & \xi t^n + x \\ 0 & 1 \end{pmatrix}, \quad \xi \in k .$$

The group G_K acts on the tree X as a group of automorphisms. We can therefore define a graph structure on the quotient F for the action of Γ on X.

THEOREM 1.2 ([S], [W]). *The quotient graph* $F = \Gamma \backslash X$ *is given by (the cosets of)*

$$\begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} \quad n \geqslant 0 ,$$

so that F *is the tree*



In fact, the vertex $\begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix}$ corresponds to n, and so if $n \geqslant 1$, its neighbor $\begin{pmatrix} t^{n+1} & x \\ 0 & 1 \end{pmatrix}$ corresponds to $n+1$ while the other q neighbors are represented by $n-1$. If $n = 0$, all neighbors correspond to 1.

2. THE OPERATOR T

Let μ be the Haar measure on G_K normalized so that $\mu(G_O) = q(q-1)$. We compute the measure of F induced from μ. Since

$$F = \Gamma \backslash X = \Gamma \backslash G_K / G_O$$

we have

$$\Gamma \backslash G_K = \cup_{s \in F} s G_O ,$$

where

$$sG_O = \{\Gamma s u \mid u \in G_O\} \subset \Gamma \backslash G_K .$$

The point measure at s will be the measure of sG_O in the quotient space $\Gamma \backslash G_K$. Now we have a correspondence

$$sG_O \simeq s^{-1}\Gamma_s s \backslash G_O ,$$

where $\Gamma_s = \Gamma \cap sG_O s^{-1}$ is the finite subgroup of Γ that stabilizes s. Thus

$$\mu(sG_O) = \frac{\mu(G_O)}{|\Gamma_s|} .$$

It is not hard to check that if $s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $|\Gamma_s| = q(q^2-1)$, while for $s = \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix}$, $n \geqslant 1$, $|\Gamma_s| = (q-1)q^{n+1}$. We therefore put mass $\frac{1}{q+1}$ at the vertex 0 and q^{-n} at the vertices $n = 1, 2, \ldots,$ so that if f and g are functions on $F = \{0, 1, 2, \ldots\}$ then

$$(2) \qquad < f, g > = \int_F f\bar{g}d\mu = \frac{1}{q+1} f(0)\bar{g}(0) + \sum_{n=1}^{\infty} f(n)\bar{g}(n)q^{-n} .$$

The algebra of operators on functions on the tree X that commute with the automorphisms of X is generated by the operator

$$(Tf)(s) = \sum_{s' \text{ is adjacent to } s} f(s')$$

(see [C2]). The operator $(q+1)I - T$ is the Laplacian on X.

If f is Γ-automorphic, and therefore can be thought of as a function on F, then T operates on f by

$$(3) \qquad (Tf)(n) = \begin{cases} qf(n-1) + f(n+1) , & \text{if} \quad n \geqslant 1 , \\ (q+1)f(1), & \text{if} \quad n = 0 . \end{cases}$$

PROPOSITION 2.1. *T is a self-adjoint operator on $L^2(F)$ with respect to the measure* μ.

Proof. If the series $\| f \|^2$ converges, then Cauchy's inequality implies that the four series in $\| Tf \|^2$ also converge. Thus T maps $L^2(F)$ into itself. Now

$$< Tf, \bar{g} > = \frac{1}{q+1}(q+1)f(1)g(0)$$

$$+ \sum_{n=1}^{\infty} (qf(n-1)g(n) + f(n+1)g(n))q^{-n}$$

$$= f(1)g(0) + \sum_{n=0}^{\infty} qf(n)g(n+1)q^{-(n+1)} + \sum_{n=2}^{\infty} f(n)g(n-1)q^{-(n-1)}$$

$$= f(1)g(0) + f(0)g(1) + \sum_{n=1}^{\infty} f(n)g(n+1)q^{-n}$$

$$+ q\sum_{n=1}^{\infty} f(n)g(n-1)q^{-n} - f(1)g(0)$$

$$= f(0)g(1) + \sum_{n=0}^{\infty} (f(n)qg(n-1) + f(n)g(n+1))q^{-n} = \langle f, T\bar{g}\rangle .$$

3. Eigenfunctions

An automorphic eigenfunction of T on X with eigenvalue λ is a function on F that satisfies

$$\lambda f(0) = (q+1)f(1) ,$$

$$\lambda f(n) = qf(n-1) + f(n+1) , \quad n \geqslant 1 .$$

If we write $u(n) = \begin{pmatrix} f(n+1) \\ f(n) \end{pmatrix}$ and normalize $u(0) = \begin{pmatrix} \lambda \\ q+1 \end{pmatrix}$, we obtain the recursion

$$u(n) = A^n u(0)$$

with

$$A = \begin{pmatrix} \lambda & -q \\ 1 & 0 \end{pmatrix} .$$

Let $x_1, x_2 = \frac{1}{2}(\lambda \pm \sqrt{\lambda^2 - 4q})$ be the characteristic roots of A and assume that $x_1 \neq x_2$, i.e., that $\lambda \neq \pm 2\sqrt{q}$. Solving the recursion we get

PROPOSITION 3.1. *The eigenfunctions on F with eigenvalue λ are the multiples of the function*

$$(4) \quad f_\lambda(n) = \begin{cases} \dfrac{1}{x_1 - x_2}(\lambda(x_1^n - x_2^n) - q(q+1)(x_1^{n-1} - x_2^{n-1})) , & \text{if } n \geqslant 1 \\ q+1 \quad \text{if} \quad n = 0 . \end{cases}$$