1. Simplices with small faces

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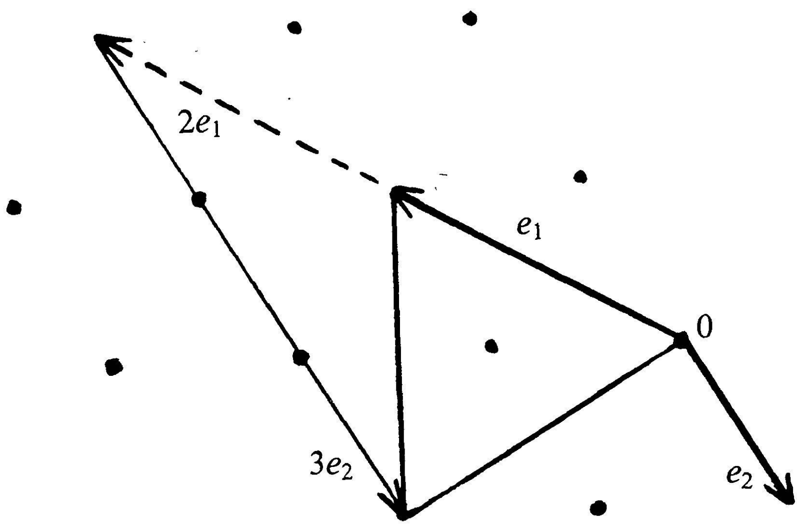
In the hexagonal lattice:



Example 0.7. Case where $n = 3, k = 2$.

Let $C = [0, 1]^3$ be the standard cube of $\mathbf{R}^3$. Let Δ be the tetrahedron defined by the vertices of the cube of which the sum of the coordinates is even. It is easy to see that Δ is affinely regular and that the linear transformation defined by

$$e_1 \mapsto -e_3 \ , \quad e_2 \mapsto e_1 + e_3 \ , \quad e_3 \mapsto e_2 + e_3$$

(where (e_1, e_2, e_3) is the standard basis of $\mathbf{R}^3$) sends Δ to the representant given in Theorem 0.5.

1. Simplices with small faces

Definition 1.1. An integral simplex S is said to have *small faces* if, for each hyperplan H containing a $(n-1)$-face of S, the vertices of S contained in H constitute an affine $\mathbf{Z}$-basis of $\mathbf{Z}^n \cap H$.

A *numerotation* of an integral simplex S is an enumeration

$$\nu = (v_0, v_1, .., v_n)$$

of the vertices of S. We will denote by S_ν the simplex S with numerotation ν. The group $\mathrm{Aff}(\mathbf{Z}^n)$ acts naturally on the set of numerated simplices and we will say that S_ν (with $\nu = (v_0, .., v_n)$) is equivalent to $S'_{\nu'}$ (with $\nu' = (v'_0, .., v'_n)$) if there exists $g \in \mathrm{Aff}(\mathbf{Z}^n)$ such that $g(v_i) = v'_i$ for all i.

The group σ_{n+1} acts on the set of numerated simplices: If $s \in \sigma_{n+1}$ is a permutation of $\{0, .., n\}$ and if S_ν is an integral simplex numerated by $\nu = (v_0, .., v_i, .., v_n)$, we define

$$s \cdot S_\nu = S_{s \cdot \nu}$$

where

$$s \cdot \nu = (v_{s^{-1}(0)}, .., v_{s^{-1}(i)}, .., v_{s^{-1}(n)}) .$$

This action of σ_{n+1} commutes with that of $\mathrm{Aff}(\mathbf{Z}^n)$, hence σ_{n+1} acts also on the equivalence classes of numerated simplices modulo $\mathrm{Aff}(\mathbf{Z}^n)$.

An integral simplex S is affinely regular if and only if the stabilizer $\mathrm{Stab}(S)$ operates transitively on the numerotations of S.

Let us recall an elementary and well-known lemma:

LEMMA 1.2. *Let* $v_1, .., v_{n-1}$ *be linearly independent vectors of* $\mathbf{Z}^n$. *Let* H *be the hyperplane of* $\mathbf{R}^n$ *generated by the* v_i's, *and suppose that* $v_1, .., v_{n-1}$ *form a basis of the sublattice* $H \cap \mathbf{Z}^n$.

Then we can complete $v_1, .., v_{n-1}$ *to a basis of* $\mathbf{Z}^n$.

Proof. See for instance the Corollary in Bourbaki, *Algèbre,* chap. VII, §4, No. 3.

From this point until the end of section 2, k will be some fixed natural integer. Let now S_ν be a numerated simplex with small faces, of volume $k/n!$. Let $\{e_1, .., e_n\}$ be the canonical basis of $\mathbf{Z}^n \subset \mathbf{R}^n$. The lemma implies that there exists $g \in \mathrm{Aff}(\mathbf{Z}^n)$ such that gS_ν has vertices

$$v_0 = 0, v_1 = e_1, .., v_{n-1} = e_{n-1}, v_n = ke_n + \sum_{i=1}^{n-1} a_i e_i$$

where the a_i's are integers. An easy calculation shows that the a_i's are well defined (mod k).

Let us associate to S_ν the element $(\alpha_1, .., \alpha_{n-1})$, where α_i is the class of a_i (mod k). This gives us a map $\rho_k \colon \Sigma_k \to (\mathbf{Z}/k\mathbf{Z})^{n-1}$, where Σ_k is the set of numerated simplices with small faces of volume $k/n!$. For $S_\nu \in \Sigma_k$, the element $\rho_k(S_\nu)$ depends only of the equivalence class of S_ν modulo $\mathrm{Aff}(\mathbf{Z}^n)$ and this allows us do define an action of σ_{n+1} on $\rho_k(\Sigma_k)$.