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THE MEAN SQUARE OF THE RIEMANN ZETA-FUNCTION ON THE LINE $\sigma = 1$

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To the memory
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ABSTRACT. Let $R(T) := \int_1^T |\zeta(1+it)|^2 dt - \zeta(2)T + \pi \log T$. We prove upper bounds for $R(T)$, $\int_1^T R(t)dt$ and $\int_1^T R^2(t)dt$.

One of the fundamental problems in the theory of the Riemann zeta-function is the evaluation of power moments, and in particular the evaluation of

$$\int_1^T |\zeta(\sigma+it)|^2 dt \,. \tag{1}$$

In view of the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, where

$$\chi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-1/2} e^{i(t+\pi/4)}\left(1+O\left(\frac{1}{t}\right)\right)$$

for $s = \sigma + it$, $t \geqslant t_0 > 0$, it transpires that the relevant range for σ in (1) is $1/2 \leqslant \sigma \leqslant 1$. A considerable amount of literature is devoted to the most important case $\sigma = 1/2$ (see e.g. Ch. 15 of [3] and [4] for a comprehensive account). For $1/2 < \sigma < 1$ fixed one has (see (8.112) of [3])

$$\int_1^T |\zeta(\sigma+it)|^2 dt = \zeta(2\sigma)T + O(T^{2-2\sigma}) \,. \tag{2}$$