4. Automatic groups : definitions and examples

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **38 (1992)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.07.2024**

Regular languages are the simplest, most important languages studied by computer scientists. They are very special — most languages (i.e., subsets of $\mathscr{A}^*$) are not regular. An example of a language over the alphabet $\mathscr{A} = \{a, b\}$ which is not regular is the set $\{a^n b^n : n \in \mathbf{Z}\}$; this follows immediately from the well-known "pumping lemma" of computer science (see [HU]). Note that $\{a^n b^m : n, m \in \mathbf{Z}\}$ is regular. The reason why $\{a^n b^n : n \in \mathbf{Z}\}$ is not regular is that a finite state automaton has no memory, and so cannot know exactly how many b's to accept after having accepted n a's.

There are other ways to define regular languages via certain grammatical operations and other machines similar to the automata described above (see [Ep1], [E *et al.*] or [HU]). Different (though equivalent) definitions of regular languages are useful in different situations, but for us the above definition will suffice.

Before giving the definition of automatic groups we will need the notion of a *two-variable padded language*. Given an alphabet $\mathscr{A}$, we can add a padding symbol $\$ \notin \mathscr{A}$ to form the alphabet $\mathscr{A} \cup \{\$\}$, and we can consider a finite state automaton W as above, but this time with labels in $(\mathscr{A} \cup \$) \times (\mathscr{A} \cup \$) \setminus (\$, \$)$. Given a pair of words $(u, v) \in \mathscr{A}^* \times \mathscr{A}^*$, say $u = u_1 \cdots u_n, v = v_1 \cdots v_m$ with $m \leqslant n$, we pad v with the symbol \$ so that the resulting words have equal length. We will say that (u, v) is accepted by W if we can read off the edges $(u_1, v_1), \ldots, (u_m, v_m), (u_{m+1}, \$), \ldots, (u_n, \$)$ and end up at an accept state of W. The set of accepted pairs (u, v) is said to be *regular over the (padded) alphabet* $\mathscr{A}$. The point of padding is that pairs of words can be read at equal speeds, even if the words have different lengths.

4. AUTOMATIC GROUPS: DEFINITIONS AND EXAMPLES

The definition of automatic group involves only finite state automata. We will later show this to be equivalent to a more geometric, and perhaps easier to understand, condition.

Let G be a group with finite generating set $\mathscr{A} = \{a_1, \ldots, a_n\}$ such that $\mathscr{A}$ actually generates G as a monoid. $\mathscr{A}$ is most often chosen as $\mathscr{A} = S \cup S^{-1}$, where S is a finite set of (group) generators for G and S^{-1} is the set of inverses of the elements of S. Notice that there is a natural map from $\mathscr{A}^*$, the free monoid on $\mathscr{A}$, to the group G which takes a word to the group element which it represents; we will denote this map by $w \mapsto \bar{w}$. G is an *automatic group* if the following conditions hold:

1. There is a language L over $\mathscr{A}$ given by a finite state automaton W such that the natural map $\pi: L \to G$ is onto; the image $\pi(w)$ is denoted by $\bar{w}$. Thus there is at least one, and perhaps even an infinite number, of words in L representing each group element. W is called the *word acceptor*, and it gives a choice of canonical forms for group elements.

2. The following (padded) languages are regular:

$$\begin{aligned} L_= &= \{(u,v):\ u,v \in L \ \text{ and } \ \bar{u} = \bar{v}\} \\ L_{a_1} &= \{(u,v):\ u,v \in L \ \text{ and } \ \bar{u} = \overline{va_1}\} \\ &\vdots \\ L_{a_n} &= \{(u,v):\ u,v \in L \ \text{ and } \ \bar{u} = \overline{va_n}\}\,. \end{aligned}$$

$L_=$ gives a FSA which checks whether two canonical forms represent the same group element; its accepting automaton $W_=$ is called the *equality checker*. L_{a_i} gives a FSA which checks whether two canonical forms represent group elements which differ by a_i (multiplied on the right); its accepting automaton W_{a_i} is called a *word comparator*.

The collection of automata $(W, W_=, W_{a_1}, \ldots, W_{a_n})$ is called an *automatic structure* for G. One should think of the word acceptor W as a way of choosing canonical forms for group elements, and the other automata of the structure as a way of relating and piecing together these canonical forms to give the group. We begin with three immediate simple examples.

Simple examples of automatic groups:

1. Finite groups are automatic. If $G = \{g_1, \ldots, g_n\}$ is a finite group of order n, we take the set $\mathscr{A} = \{g_1, \ldots, g_n\}$ as monoid generating set, and let $L = \mathscr{A}$. Then L is a regular language since finite sets are regular (the automaton is an n-segment star with labels g_i on the edges). Since the sets $L_=, L_{g_1}, \ldots, L_{g_n}$ are finite, hence regular, G is automatic.

2. The infinite cyclic group $\mathbf{Z} = \langle a \rangle$ is automatic. Figure 3 gives the automatic structure for $\mathbf{Z}$ with respect to the generating set $\mathscr{A} = \{a, A = a^{-1}\}$.

3. The free group on two generators $F(a,b)$ is automatic. The word acceptor given in figure 2, which accepts reduced words in $F(a,b)$, is part of an automatic structure. The reader is invited to construct the equality checker and the comparator automata.

An automatic group may have many automatic structures, even for a fixed generating set; the point is to find an automatic structure which is natural and easy to understand. It has been shown that if G is automatic with respect to one finite set of generators, then it is automatic with respect to any other finite set of generators ([E *et al.*]).

There is another definition of automatic group which is equivalent to the definition we have given. This second definition is more geometric than the first, and so it is often useful in proving that groups arising in geometric situations are automatic. We shall prove the equivalence of the two definitions since the proof gives a taste of the interplay between the geometry and the finite state automata.

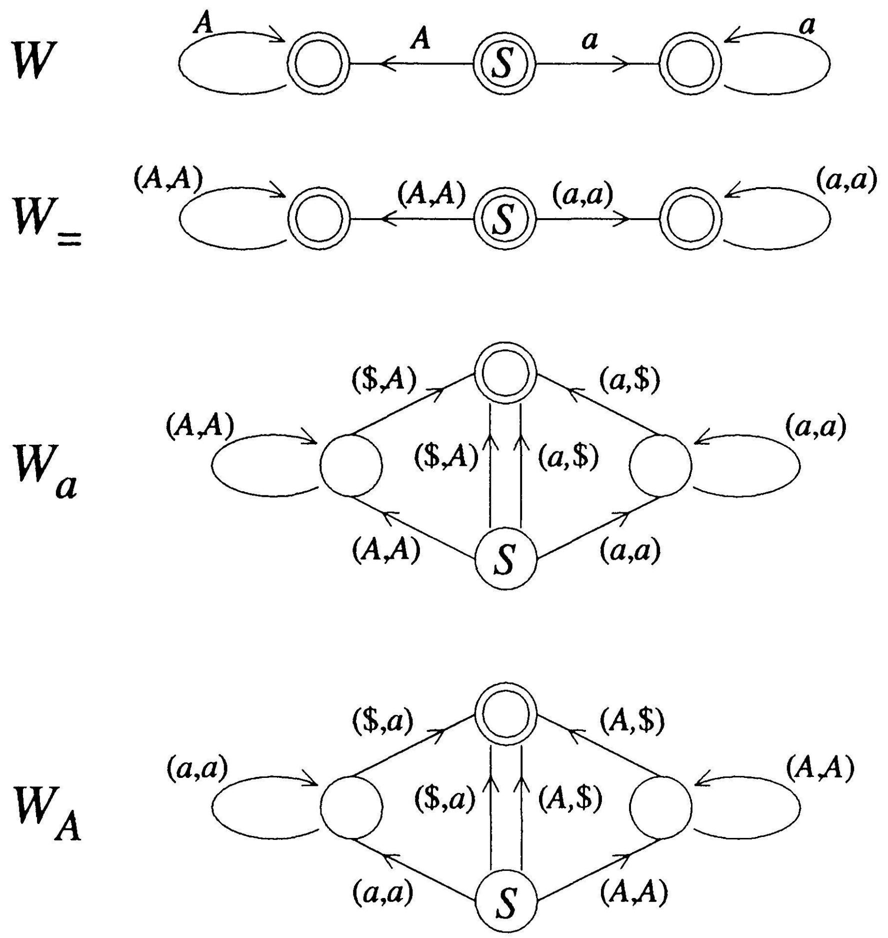

FIGURE 3

An automatic structure for $\mathbf{Z} = < a, A = a^{-1} >$

Recall that a path u in $\Gamma_S(G)$ can be thought of as a map u: $[0, \infty) \to \Gamma_S(G)$, where $u(t)$ is the element of G given by the first t letters of u if t is less than the length of u, and $u(t) = \bar{u}$ if t is greater than or equal to the length of u (recall that $\bar{u}$ is the element of G represented by the word u in the free group on S). Two paths u and v in $\Gamma_S(G)$ are said to satisfy the *k-fellow traveller property* if $d_{\Gamma_{S(G)}}(\overline{u(t)}, \overline{v(t)}) \leqslant k$ for all $t \geqslant 0$.

PROPOSITION 1. *A group G is automatic if and only if the following properties hold:*

1. G has a word acceptor W with regular language $L(W)$ over some finite monoid generating set $\mathscr{A}$, as in condition 1 of the definition of automatic group. Recall that the natural map $L(W) \to G$ is required to be onto, and that the image of a word w is denoted by $\bar{w}$.

2. There is a constant k such that if $u, v \in L(W)$ represent elements of G which are distance 1 apart in $\Gamma_{\mathscr{A}}(G)$, then the paths u and v satisfy the k-fellow traveller property.

Proof. If G is automatic with monoid generating set $\mathscr{A} = \{a_1, \ldots, a_n\}$, let c be an integer greater than the maximum number of states in any of the word comparators $W_{a_1}, \ldots, W_{a_n}$. Suppose $\bar{u}$ and $\bar{v}$ are a distance 1 in $\Gamma_{\mathscr{A}}(G)$, so that $u, v \in L$ differ by some a_i, say $u = va_i$. Let $s(t)$ denote the state W_{a_i} is in after reading the (possibly padded) prefixes $u(t)$ and $v(t)$. Then clearly there is a path in W_{a_i} from $s(t)$ to an accept state, and this path must have length less than c. Note that the "path to an accept state" in W_{a_i} gives a pair of paths in $\Gamma_{\mathscr{A}}(G)$ from $\overline{u(t)}$ and $\overline{v(t)}$ to points in $\Gamma_{\mathscr{A}}(G)$ which differ by the generator a_i; each of these paths must have lengths (in the Cayley graph) less than c. From this it follows easily that $d_{\Gamma_S(G)}(\overline{u(t)}, \overline{v(t)}) \leqslant 2(c-1) + 1 = 2c - 1$ for all t (see figure 4). This bound holds uniformly for all such paths.

Now suppose G satisfies conditions (1) and (2) of the hypothesis. We'll build a finite state automaton *Diff* which keeps track of how two paths differ

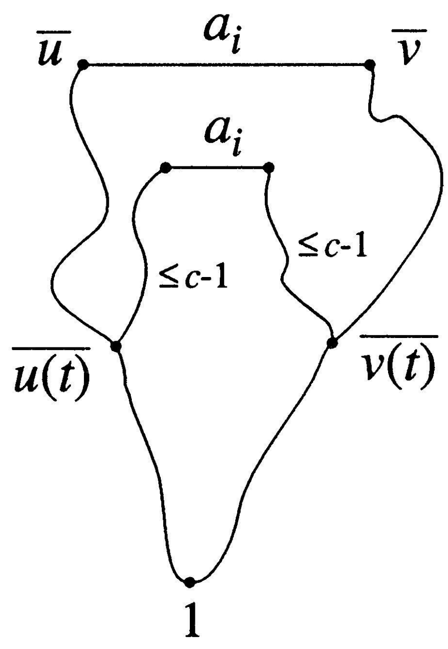

FIGURE 4

Proof of the k-fellow traveller property for an automatic group over time, and we'll use *Diff* to build the W_{a_i}. If S and s_0 are the state set and start state of the word acceptor W, and if B_k is the set of group elements of distance $\leqslant k$ from the identity $e \in G$ (i.e. the k-ball in $\Gamma_{\mathscr{A}}(G)$ around e), let the state set S' of *Diff* be $S \times S \times B_k$, and let the start state of *Diff* be (s_0, s_0, e). If *Diff* is in state (s_1, s_2, g) and the letter (x, y) in the associated padded alphabet is read, then *Diff* goes into the state $(t_1, t_2, x^{-1}gy)$, where x (resp. y) takes state s_1 (resp. s_2) of W to state t_1 (resp. t_2) of W (with the

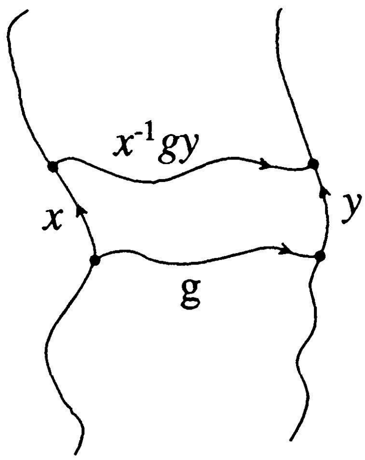

FIGURE 5

The idea behind the finite state automaton Diff

padding symbol not changing the states); except if t_1 or t_2 is a fail state of W or if $x^{-1}gy \notin B_k$, in which case *Diff* goes into fail state (don't draw an arrow labelled (x,y) coming out of (s_1, s_2, g)). Now *Diff* is a FSA which keeps track of whether two paths are accepted by W, and also keeps track of how far away the two paths are (see figure 5). The automata W_{a_i} may then be defined by taking the FSA *Diff* with accept states of the form (s_1, s_2, a_i), where $s_1, s_2 \in S$. $W_=$ is given by *Diff* with accept states of the from (s_1, s_2, e), where $s_1, s_2 \in S$. □

Proposition 1 shows that an automatic structure for a group G with generating set $\mathscr{A} = \mathscr{A}^{-1}$ is determined by the regular set $L \subseteq \mathscr{A}^*$; the word acceptor for L and the comparator automata must exist, but they need not be given explicitly. Most often one shows that a certain set of words L, such as the set of geodesics in the Cayley graph, is regular, and that the k-fellow traveller property is satisfied for some k. The proof of Proposition 1 also shows that the entire Cayley graph $\Gamma_{\mathscr{A}}(G)$ is determined by the k-ball around the identity; the "linear recursion" given by the automata of the automatic structure may be used to knit together copies of this k-ball to obtain $\Gamma_{\mathscr{A}}(G)$.

Automatic groups encompass a large class of examples under one theory. We provide a list of the most well-known examples, others are being discovered at a rapid pace. The proof that a given class of groups is automatic usually involves doing quite a bit of geometry in spaces on which that class of groups acts in a geometric way (e.g., cocompactly by isometries); hence the proofs of the facts below have a strongly geometric flavor.

MAIN EXAMPLES OF AUTOMATIC GROUPS:

1. *Finite groups.*

2. *Abelian groups.*

3. *Negatively curved groups.* Negatively curved groups (sometimes called 'hyperbolic groups') are those groups whose Cayley graphs have uniformly thin triangles. These groups have been studied extensively in the last ten years (see [Gr, Ca2, Gh, CDP, GdlH] for surveys). Examples of negatively curved groups include finite groups, free groups, cocompact discrete groups of hyperbolic isometries (more generally, fundamental groups of compact manifolds with strictly negative sectional curvatures), and small cancellation groups satisfying the usual metric small cancellation conditions ([LS]). Gromov has claimed that, in some combinatorial sense, "most" groups are negatively curved. The proof that negatively curved groups are automatic is essentially contained in Cannon's original paper ([Ca1]), although it is of course not couched in those terms; the language L of normal forms consists of the set of geodesic words. Negatively curved groups were the first and are still the most important examples of automatic groups. Automatic groups are much more general than negatively curved groups; for example, negatively curved groups cannot have a subgroup isomorphic to $\mathbf{Z} \times \mathbf{Z}$. In fact, any Seifert fiber space over a closed surface has an automatic fundamental group ([E *et al.*]), but most of these spaces do not even admit metrics of non-positive curvature.

4. *Non-metric small cancellation groups.* Groups with a presentation satisfying the weaker, non-metric small cancellation hypothesis are not, in general, negatively curved. However, Gersten and Short have shown that such groups are automatic ([GS1, GS2]). In some sense the theory of automatic groups unifies and supercedes small cancellation theory.

5. *Many Coxeter groups.* Many Coxeter groups are automatic. It still seems to be an open question whether all Coxeter groups are automatic. It is quite enjoyable to construct explicitly automatic structures for reflection groups in the euclidean and hyperbolic planes, such as the group of reflections

in the sides of a right-angled pentagon in the hyperbolic plane; the reader is encouraged to do so.

6. *Most three-manifold groups.* The situation for compact three-manifolds is pretty well understood. Epstein and Thurston have shown ([E *et al.*]) that if M is a compact three-manifold which satisfies Thurston's Geometrization Conjecture, then $\pi_1(M)$ is automatic if and only if none of the pieces in its decomposition along spheres and tori is modelled on Nilgeometry or Solvgeometry. That is, if $M_1, \ldots, M_k$ are compact three-manifolds whose interiors are modelled on one of the eight three-dimensional geometries, and if M is the compact, connected three-manifold formed from the M_i by connected sum, disk sum and identifying boundary tori in pairs, then $\pi_1(M)$ is automatic if and only if none of the M_i is closed and modelled on Nilgeometry or Solvgeometry. Another approach to this, via theorems on the automaticity of graphs of groups, is given by Shapiro ([S]).

7. *Geometrically finite groups.* Epstein has shown ([E *et al.*]) that every geometrically finite hyperbolic group is automatic; in particular, fundamental groups of hyperbolic link complements are automatic. This is useful since most link complements have a hyperbolic structure. The main part of Epstein's proof involves figuring out what the quasi-geodesics (see below) are in the universal cover of a finite volume hyperbolic manifold with its cusps cut off.

8. *Braid groups.* Thurston has shown ([E *et al.*]) that the braid group on n strands is automatic (for each $n \geqslant 1$), which also shows that the mapping class group of the $(n+1)$-punctured sphere is automatic. This work explores several algorithmic aspects of the braid group. Thurston has conjectured that the mapping class groups of all hyperbolic surfaces are automatic.

The property of being automatic is closed under direct product, free product and free product with amalgamation over a finite subgroup. If H is a finite index subgroup in G, then H is automatic if and only if G is. These closure properties give many more examples of automatic groups; in particular, cocompact discrete groups of Euclidean isometries are automatic since they contain abelian subgroups of finite index by Bieberbach's Theorem.

Although a wide variety of examples are automatic, this class of groups is very special, much more so than, say, groups with solvable word problem. To show that a group is not automatic seems difficult, for how does one show that "There does not exist any regular language such that..."? However, techniques for showing that certain groups are not automatic have been developed; most of these involve isoperimetric inequalities in groups. We refer the reader to [E *et al.*], [Ge1], and [GS3] for details.

EXAMPLES OF GROUPS THAT ARE NOT AUTOMATIC:

1. *Infinite torsion groups.* The mere existence of such groups is far from trivial, so it is not very disconcerting that such groups are not automatic (it is, perhaps, heartening). That infinite torsion groups are not automatic follows immediately from the well-known "pumping lemma" for finite state automata (see [Gi] for the proof).

2. *Nilpotent groups.* Finitely generated nilpotent groups which do not contain an abelian subgroup of finite index are not automatic. This was first proved by Holt. For example the three-dimensional Heisenberg group $H_3 = < a,b,c: [a,b] = c, [a,c] = 1 = [b,c] >$, the simplest non-abelian nilpotent group, has a cubic isoperimetric function (see property 7 below) and so is not automatic. The fact that nilpotent groups are not automatic is a bit surprising and annoying, considering the fact that nilpotent groups are quite common and have an easily solved word problem.

3. $SL_n(\mathbf{Z}), n \geqslant 3$. Note that $SL_2(\mathbf{Z})$ contains a free subgroup of index six, and so is automatic. The proof that $SL_n(\mathbf{Z}), n \geqslant 3$ is not automatic involves finding a contractible manifold on which $SL_n(\mathbf{Z})$ acts with compact quotient, and showing that a higher-dimensional isoperimetric inequality is not satisfied by that space. The search for this manifold involves the study of the symmetric space $SL_n(\mathbf{R}) / SO_n(\mathbf{R})$.

4. *Baumslag-Solitar Groups.* The group $G_{p,q} = < x, y: yx^p y^{-1} = x^q >$ is not automatic unless $p = 0$, $q = 0$ or $p = \pm q$. These groups provide examples of groups which are not automatic but are *asynchronously automatic* (see [BGSS, E *et al.*]).

5. HYPERBOLIC GROUPS ARE AUTOMATIC

It is most often the case that proving that a group G is automatic requires doing quite a bit of geometry in a space on which G acts in a geometric way. As an example we prove the result of Cannon that cocompact discrete groups of hyperbolic isometries are automatic; in fact we show this more generally for fundamental groups of compact manifolds with (not necessarily constant) strictly negative sectional curvatures.

A path $\alpha: [a,b] \to X$ in a metric space X is a *quasi-geodesic* if it is a geodesic up to constants; that is, there exists a K such that

$$1 / K(t_2 - t_1) - K < d_X(\alpha(t_1), \alpha(t_2)) < K(t_2 - t_1) + K$$