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7. *Quadratic isoperimetric function.* We shall not discuss isoperimetric functions in groups here; the reader may consult [Ge1, Sh, E *et al.*]. Isoperimetric functions in groups are extremeley interesting, and have become quite important in combinatorial group theory and geometry; related concepts have recently proven useful in the study of three-manifolds ([Ge2, St]). Gromov showed that the negatively curved groups are precisely those which have a linear isoperimetric function. Automatic groups satisfy a quadratic isoperimetric function, but are not characterized by this property. Thurston (unpublished) has shown that the five-dimensional Heisenberg group has a quadratic isoperimetric function but is not automatic.

It is possible to get a feel for the breadth and unifying power of the theory of automatic groups by matching up groups in the list of examples with properties from the list just given. For instance, automatic groups give a uniform quadratic time solution to the word problem for fundamental groups of compact negatively curved manifolds, most Coxeter groups, and the braid groups (previously known algorithms for the braid groups never discussed speed, and seem to be much slower). The reader may wish to contemplate other "theorems" obtained by matching pairs in the lists.

7. Related topics, open problems, and a vision of the future

The field of automatic groups (and related topics) is still quite young; accordingly, there are many open questions which are interesting, easy to state, and perhaps not so difficult for a newcomer to think about. Listed below are a few personal favorites. For other open questions, the reader is encouraged to dive into the references given at the end of this paper, in particular [Ge3].

Some open problems:

1. Prove that the mapping class groups of hyperbolic surfaces are automatic. As a (perhaps) easier question, show that these groups satisfy a quadratic isoperimetric inequality ([Ge1, E *et al.*]).

2. Are cocompact lattices in $SL_3(\mathbf{R})$ automatic? Note that $SL_3(\mathbf{Z})$ is a lattice in $SL_3(\mathbf{R})$ which is not cocompact and not automatic. There is a p-adic analog to this question which has been solved ([GS1]). Find examples of other arithmetic groups which are or are not automatic. So far not much seems to be known for arithmetic groups, except for a result of Gersten and Short ([GS3]) which shows that $SL_2(\mathscr{O})$, with $\mathscr{O}$ a real quadratic number field, is

not biautomatic (see below). Note that for some rings of algebraic integers, such as $\mathcal{O} = \mathbf{Z}[i]$, the group $SL_2(\mathcal{O})$ is the fundamental group of a three manifold which is automatic.

3. Are fundamental groups of compact, non-positively curved manifolds automatic? The answer to this question is probably "no" (see 9 below).

4. A simple question whose answer has eluded everyone: If $G \times H$ is automatic, is G automatic? The corresponding statement for free products is true ([BGSS]).

5. Negatively curved groups have a well-defined "boundary at infinity" which gives a great deal of information about such groups (see any of the surveys on negatively curved groups cited above). Is there a corresponding theory for automatic groups?

6. Explore the effect of changing generators and automatic structures on the constant k of the k-fellow traveller property. What is the best constant you can get for a specific example (e.g. a surface group)? For a given group G, what is the minimal number of states of a word acceptor which is part of an automatic structure with unique representatives for G?

7. Study the quasi-convex subgroups of automatic groups (see [GS3]), as well as other geometric properties which have algorithmic consequences. Work this out explicitly for fundamental groups of three-manifolds.

8. Does every automatic group have a rational counting function? This is true for automatic groups where the language of accepted words consists of geodesics. Explore analytic properties of these functions in special cases (Cannon and others have done this for several examples).

9. A group is *combable* if there is a section $\sigma: G \to \mathcal{A}^*$ of the natural map $\pi: \mathcal{A}^* \to G$ which satisfies the k-fellow traveller property for some k for all paths. Automatic groups are simply combable groups whose image $\sigma(G) \subseteq \mathcal{A}^*$ is a regular language. Much of the theory of automatic groups has been generalized to combable groups ([Sh]). Combability is also a very natural condition to look at when studying geometry, in particular the geometry of nonpositive curvature. Find an example of a combable group which is not automatic. A good place to look (according to Thurston) might be at a cocompact group of isometries of $\mathbf{H}^2 \times \mathbf{H}^2$ which does not have a product of surface groups as a subgroup of finite index. This would also show that the property of being automatic is not a so-called "geometric invariant", i.e., a quasi-isometry invariant, but depends on more combinatorial properties of the group.

10. Are automatic groups residually finite? Gersten ([Ge4]) has recently found a combable group that is not residually finite. Which automatic groups admit a faithful linear representation (such groups are residually finite)? More generally, what bearing does the automatic structure have on the representation theory of an automatic group?

11. A notion stronger than automatic is that of *biautomatic,* where one is also supplied with word comparators W'_{a_i} for multiplication by a_i on the left. Much more is proven about biautomatic groups than automatic groups (e.g. biautomatic groups have solvable conjugacy problem), although it is not known whether every automatic group is biautomatic. There seems to be a deep theory of subgroup structure for biautomatic groups, as has been developed by Gersten and Short ([GS3]). Carry over the theory of biautomatic groups to automatic groups, in particular solve the conjugacy problem for automatic groups and also find theorems which put some constraint on what the subgroups of an automatic group can be (see [Ge1, GS3]). Even better, determine whether every automatic group is in fact biautomatic.

12. Generalize the entire theory of automatic groups by using machines which are more complicated than finite state automata (see below).

Thurston has envisioned a program of studying algorithmically groups that arise naturally in geometry and topology. Automatic groups are the first stage of this program. One idea is to relativize the theory by replacing the states of the automata by black boxes which could do computations in nilpotent groups in order to study groups which are automatic (or hyperbolic) "relative to" certain nilpotent subgroups; examples being fundamental groups of (non-compact) finite volume complex hyperbolic manifolds. Another direction might be to replace finite state automata by more complicated machines. In the hierarchies of languages and machines studied by complexity theorists (e.g. the Chomsky Hierarchy), regular languages and finite state automata are always at the bottom of the ladder; in fact, regular languages may be characterized by the fact that it takes zero-space of an (off-line) Turing machine to recognize them (see [HU]). More complicated machines should allow us to do computations (such as solving the word problem efficiently) in more complicated groups; the geometry of the group dictating the nature of the machine. The possibilities seem limitless.