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TOEPLITZ SEQUENCES, PAPERFOLDING, TOWERS OF HANOI AND PROGRESSION-FREE SEQUENCES OF INTEGERS

by Jean-Paul ALLOUCHE and Roland BACHER

ABSTRACT. What is the relationship between folding a piece of paper, moving disks in the classical tower of Hanoi algorithm and searching for minimal sequences of integers having no p terms in arithmetic progression? Our aim is to show how the Toeplitz sequences introduced by Jacobs and Keane in [15] allow us to give *(inter alia)* a unified description of the preceding problems. We give moreover some connections between Toeplitz sequences and q-automatic sequences.

1. TOEPLITZ SEQUENCES

In [15], (see also [21]), Jacobs and Keane defined the notion of Toeplitz sequence: they wanted to construct "explicit" sequences giving rise to strictly ergodic systems. They proved moreover that the unique invariant measure attached to such a sequence has a discrete rational spectrum. Roughly speaking a Toeplitz sequence is obtained by successive insertions of periodic sequences into the "holes" of a given periodic sequence, (a precise definition is given below). This construction was inspired by a device used by Toeplitz [28] for building explicitly almost periodic real functions. The method of Jacobs and Keane has since been used by many people working in ergodic theory (see for instance [29], [16] and [25], see also [14] and its impressive bibliography). We now give the definition of a Toeplitz sequence (compare with [15], [16], [14] and [29]):

Let $\Gamma = \{a_1, \cdots, a_r, \omega\}$ be an alphabet (finite set) with a "marked" letter ("hole") ω. If $B = (B(k))_{k \geqslant 0}$ is a sequence with values in Γ, we define a transformation $T_B: \Gamma^{\mathbf{N}} \rightarrow \Gamma^{\mathbf{N}}$ as follows: for any sequence $C = (C(k))_{k \geqslant 0}$ with values in Γ, let $h_0 < h_1 < \cdots$ be the increasing sequence (which might be finite or even empty) of those integers h for which $C(h) = \omega$. Then one defines

$$T_B C(j) = C(j) \quad \text{if} \quad C(j) \neq \omega ,$$
$$T_B C(h_k) = B(k) \quad \text{for every } k .$$

Suppose we are now given a sequence of periodic sequences $B_0, B_1, \cdots, B_k, \cdots$ with values in Γ, and such that the zeroth value of each B_j is not equal to ω. Writing T_j instead of T_{B_j}, we then define a sequence of periodic sequences as follows:

$$A_0 = B_0$$
$$A_1 = T_1(A_0)$$
$$A_2 = T_2(A_1) \quad = T_2(T_1(A_0))$$
$$\cdots$$
$$A_{k+1} = T_{k+1}(A_k) = T_{k+1}(T_k(\cdots(T_1(A_0))\cdots)) .$$

As k goes to infinity the sequence A_k tends to a limit A with values in $\Gamma - \{\omega\}$ (the existence of this limit, for the topology of simple convergence, is left to the reader): such a sequence is called a *Toeplitz sequence.*
An alternative (equivalent) definition of a Toeplitz sequence is given in [29]:

A is a Toeplitz sequence if and only if one has

$$\forall n \in \mathbf{N} \quad \exists p \in \mathbf{N}^* \quad \forall n' \equiv n \pmod p \quad A(n') = A(n) .$$

In what follows we first suppose that the set Γ is not necessarily a finite set; second, we restrict ourselves to the case where the sequence $B_0, B_1, \cdots$ has the following form: there exist a periodic sequence B with values in Γ such that $B(0) \neq \omega$ and a function f from Γ to Γ with $f^{-1}(\omega) = \{\omega\}$, such that

$$\forall k \geqslant 0 \quad B_k = f^{(k)}(B) ,$$

where $f^{(k)}$ is the k^{th} iterate of the function f and $f^{(k)}(B)$ is the termwise image of the sequence B under $f^{(k)}$; the resulting Toeplitz sequence

$$A = \lim_{k \to \infty} T_k(\cdots T_2(T_1(B))\cdots)$$

$$(\text{where } T_k = T_{B_k} = T_{f^{(k)}(B)})$$

will be called the *Toeplitz transform* of (B, f) and denoted by $Tt(B, f)$.

Example. Let B be the sequence $(0\omega 1\omega)^\infty$, and let f be defined on $\{0, 1, \omega\}$ by $f(0) = 1$, $f(1) = 0$, $f(\omega) = \omega$, then one has:

$$B_0 = (0\omega 1\omega 0\omega 1\omega \cdots)\ (= B)\ ,$$
$$B_1 = (1\omega 0\omega 1\omega 0\omega \cdots)\ ,$$
$$B_2 = B_0\ ,$$
$$\ldots$$
$$A_0 = (0\omega 1\omega 0\omega 1\omega \cdots)\ ,$$
$$A_1 = (011\omega 001\omega \cdots)\ ,$$
$$A_2 = (0110001\omega \cdots)\ ,$$
$$\ldots\ .$$

Note that if Γ is finite, and f one-to-one, such a sequence $Tt(B,f)$ can also be obtained by replacing B by a sequence of greater period and f by *id*.

We now give four examples of Toeplitz transforms in (apparently) unrelated domains.

2. Paperfolding sequences and Toeplitz transforms

In [23] and [22] Prodinger and Urbanek study the Toeplitz transform of $((0\omega 1\omega)^\infty, id)$ and of $((0\omega 1\omega 1\omega 0\omega)^\infty, id)$. They prove that these sequences do not have arbitrarily long squares (a sequence A contains a square of length $2k$ if there exists an index j such that $A(j+n) = A(j+n+k)$ for every n between 0 and $k-1$). Dekking already noticed in [10] that the first sequence is nothing but the regular paperfolding sequence (see [9], [18], [20], [17]), which is obtained by repeatedly folding a piece of paper, and we obtained in [1] the same result as Prodinger and Urbanek for the general paperfolding sequences. Let us give here two simple examples:

PROPOSITION. *Let B be the sequence $B = (0\omega 1\omega)^\infty$ and let f be defined by $f(0) = 1$, $f(1) = 0$ and $f(\omega) = \omega$. Then*

the sequence $Tt(B, id)$ is the regular paperfolding sequence,

the sequence $Tt(B, f)$ is the alternate paperfolding sequence.

Proof. It follows from instance from [18] (after replacing 1's by 0's and -1's by 1's) that the regular paperfolding sequence R and the alternate paperfolding sequence A are given by

$$R(2^k(2m+1)-1) = \frac{(1-(-1)^m)}{2} \quad \forall k, m \geqslant 0 .$$

$$A(2^k(2m+1)-1) = \frac{(1-(-1)^{k+m})}{2} \quad \forall k, m \geqslant 0 .$$

Let U and V be the sequences defined by

$$U = Tt((0\omega\, 1\omega)^\infty, id) ,$$

$$V = Tt((0\omega\, 1\omega)^\infty, f) .$$

A straightforward computation gives

$$U(2n) = \frac{(1-(-1)^n)}{2} \quad \forall n \geqslant 0 ,$$

$$U(2n+1) = U(n) \quad \forall n \geqslant 0 .$$

Hence

$$U(2^k(2m+1)-1) = U(2(2^{k-1}(2m+1)-1)+1)$$

$$= U(2^{k-1}(2m+1)-1) = \cdots = U(2m) = \frac{(1-(-1)^m)}{2} .$$

This proves that $U = R$.

In the same way one has

$$V(2n) = \frac{(1-(-1)^n)}{2} ,$$

$$V(2n+1) = 1 - V(n) .$$

Hence

$$V(2^k(2m+1)-1) = V(2(2^{k-1}(2m+1)-1)+1) = 1 - V(2^{k-1}(2m+1)-1)$$

$$= \cdots = \left\{ \begin{array}{ll} V(2m) & \text{if } k \text{ is even,} \\ 1 - V(2m) & \text{if } k \text{ is odd,} \end{array} \right\} = \frac{(1-(-1)^{k+m})}{2} ,$$

and finally $V = A$.

3. Iteration of continuous functions and Toeplitz transforms

When iterating a unimodal (i.e. increasing then decreasing) and continuous map of the interval $[0, 1]$, say F_μ, depending on the parameter μ, one knows that, assuming certain properties of the map $\mu \to F_\mu$, a Feigenbaum doubling cascade phenomenon occurs (see [7] for instance): when the parameter increases, the function has first an attractive fixed point for $\mu_0 \leqslant \mu < \mu_1$, then an attractive cycle of length 2 for $\mu_1 \leqslant \mu < \mu_2$, then an attractive cycle of length 4, ... There is a "first" value of the parameter μ_∞ for which a "chaotic" behaviour appears, and this value is the limit of the sequence $(\mu_n)_n$. This sequence grows roughly like a constant term plus a geometric progression C^n, where the constant C is universal, provided that the functions F_μ are smooth enough. This constant is called the Feigenbaum constant.

The orbit of the point 1 under F_{μ_∞} can be coded by a universal binary sequence A (even in cases where the Feigenbaum constant does not appear). This sequence $A = (A(n))$ is defined as 0 if $F^{(n)}_{\mu_\infty}(1)$ is smaller than the point where F_{μ_∞} takes its maximum, and 1 if $F^{(n)}_{\mu_\infty}(1)$ is larger than this value, and does not depend on the family of functions (F_μ). Moreover it has been noticed in [3] that the sequence A is related to the Prouhet-Thue-Morse sequence C (see [6] and its bibliography) by

$$C(n+1) = \sum_{0 \leqslant j \leqslant n} A(j) \text{ modulo } 2 .$$

(Let us recall that C is the fixed point beginning by 0 of the 2-substitution $0 \to 01, 1 \to 10$.)

Actually as noticed in [24], A is the fixed point of the 2-substitution

$$1 \to 10, \quad 0 \to 11 .$$

PROPOSITION. *Let f be defined by $f(0) = 1, f(1) = 0, f(\omega) = \omega$, then the sequence A is the Toeplitz transform of $((1\omega)^\infty, f)$.*

Proof. As the fixed point of the 2-substitution $1 \to 10$, $0 \to 11$, the sequence A can be recursively defined by

$$A(2n) = 1 , \quad A(2n+1) = 1 - A(n) .$$

Remark. The relation between C and A can also be written

$$A(n) = C(n) + C(n+1) \text{ modulo } 2 .$$

If, instead of C one takes a "generalized" Morse sequence C', and if one defines

$$A'(n) = C'(n) + C'(n+1) \text{ modulo } 2\,,$$

then A' is also a Toeplitz sequence, as proved in [16].

4. TOWERS OF HANOI AND TOEPLITZ SEQUENCES

The tower of Hanoi puzzle consists of three vertical pegs and of N circular disks of different diameters stacked in decreasing order on the first peg. At each step one may transfer the topmost disk from a peg to a different peg according to the rule: no disk is allowed to be on a smaller one. The game ends when all the disks are stacked on the second or third peg.

The sequence of moves for the classical (minimal) Hanoi tower algorithm can be generated in a very easy way as it is 2-automatic (see [4] and section 6), which essentially means that the k^{th} move can be predicted by a machine with bounded memory. More precisely number the pegs as I, II, III and define a (respectively b, c) to be the move which takes the topmost disk from peg I (respectively II, III) and puts it on peg II (respectively III, I). Let $\bar{a}, \bar{b}, \bar{c}$ be the respective opposite moves. Then the sequence of moves for N disks is the prefix of length $2^N - 1$ of an infinite sequence U which is 2-automatic. Moreover the following proposition is proved in [4]:

PROPOSITION. *The infinite sequence of moves U is equal to the Toeplitz transform of* $((a\bar{c}b\omega\, c\bar{b}a\omega\, b\bar{a}c\omega)^\infty, id)$.

Note that, keeping the notations of [4], the sequence U is indexed by $1, 2, \cdots$ and not by $0, 1, 2, \cdots$ as the sequences above.

5. PROGRESSION-FREE SEQUENCES AND TOEPLITZ SEQUENCES

The question of finding a sequence of integers without arithmetic progressions of given length has been intensively studied (see its history in [14] and the included bibliography). In particular what is the "minimal" increasing sequence having this property?

One knows that, if k is a prime number, the minimal sequence of integers without any arithmetic progression of k terms is exactly the increasing sequence of the integers without the digit $k-1$ in their base k expansion (cited in [11], [26] and [12]). But nothing is known for the case where k is not prime (see [12]).

Let us define for every integer $k \geqslant 3$, $(U_k(n))$ as the increasing sequence of the integers without the digit $k-1$ in their base-k expansion. It is not difficult to obtain:

$$(*) \qquad \forall j \in [0, k-2]\,, \qquad U_k((k-1)n+j) = kU_k(n) + j\,.$$

If one considers the sequence of first differences of, say U_3, one obtains the sequence:

$$1 \quad 2 \quad 1 \quad 5 \quad 1 \quad 2 \quad 1 \quad 14 \quad 1 \quad 2 \quad 1 \quad 5 \quad 1 \quad 2 \quad 1 \quad \cdots$$

This sequence resembles somewhat the paperfolding sequence, (except that it takes infinitely many values), which gives the idea of the following easy proposition:

PROPOSITION. *Let k be an integer greater than or equal to 3, define the sequence $(U_k(n))$ by $(*)$. Let $D_k(n) = U_k(n+1) - U_k(n)$. Finally let g_k be defined on $\mathbf{N} \cup \{\omega\}$ by $g_k(x) = kx - k + 2$ if x is in $\mathbf{N}$ and $g_k(\omega) = \omega$.*

Then

$$D_k = Tt((1^{k-2}\omega)^\infty, g_k)\,,$$

(see notations in paragraph 1).

Proof. From the definition of U_k, one has

$$D_k((k-1)n+j) = 1 \quad \text{for every } j \text{ in } [0, k-3] \text{ and every integer } n\,,$$

$$D_k((k-1)n+k-2) = kD_k(n) - (k-2) = g_k(D_k(n)) \quad \text{for every integer } n\,.$$

Remark. For a very curious occurrence of the sequence U_k see [19].

6. Miscellaneous questions

In this paragraph we first give some other examples of naturally occurring Toeplitz sequences. Second we shall study the connections with automatic sequences.

1) Among other examples of Toeplitz transforms let us give three natural sequences:

— Let p be a prime number, and $\nu_p(n)$ be the highest power of p dividing n. Let $U(n) = \nu_p(n+1)$, and let f be the function defined over $\mathbf{N} \cup \{\omega\}$ by $f(x) = x + 1$ for every integer x and $f(\omega) = \omega$. Then: $U = Tt((0^{p-1}\omega)^\infty, f)$.

— Define, for $n \geqslant 1$, $Q(n) = 1$ if n is the sum of three squares, and $Q(n) = 0$ otherwise. Then $Q = Tt((111\omega 110\omega)^\infty, id)$.

— Let $C(n)$ be the van der Corput sequence (see [8]), used in the theory of distribution modulo 1 and defined by:

$$\text{if } n = \sum_{i \geqslant 0} b_i(n)2^i, \text{ where } b_i \text{ is 0 or 1, then } C(n) = \sum_{i \geqslant 0} b_i(n)2^{-i-1}.$$

Let $V(n) = C(n+1) - C(n)$ be the difference sequence of C. Let finally h be defined on the rational numbers by $h(x) = \dfrac{x-1}{2}$ and at ω by $h(\omega) = \omega$.

Then one has $V = Tt\left(\left(\frac{1}{2}\,\omega\right)^\infty, h\right)$.

The first and second proofs are left to the reader; a hint for the third proof is that $C(2n) = \dfrac{C(n)}{2}$ and $C(2n+1) = \dfrac{(1 + C(n))}{2}$.

2) The interested reader can find in [27] a beautiful continued fraction expansion for $\psi(1)$ where ψ is the Carlitz exponential function for $F_2[T]$, and he will certainly recognize a Toeplitz transform hidden in this expansion.

3) Actually all sequences given so far are either q-automatic (see [6] or [2]) or q-regular (see [5]). Let us recall that a sequence $(U(n))$ is said to be q-automatic if its q-kernel (i.e. the set of subsequences $n \to U(q^k n + r)$ of the sequence U, where $k \geqslant 0$ and $0 \leqslant r \leqslant q^k - 1$) is finite. A sequence U with values in a noetherian ring R is said to be q-regular if its q-kernel spans an R-module of finite type.

If one takes the regular paperfolding sequence A, it is not hard to check that ist 2-kernel is finite and equal to $\{A, (01)^\infty, (0)^\infty, (1)^\infty\}$, hence the sequence A is 2-automatic.

In which case does a periodic sequence B with values in a finite alphabet give rise to an automatic Toeplitz transform? We give the following answer to this question:

THEOREM. *Let B be a periodic sequence of period T with values in $\Gamma = \{a_1, a_2, \cdots, a_r, \omega\}$, such that $B(0) \neq \omega$. Denote by d the cardinality of the set $\{h \in [0, T-1] \mid B(h) = \omega\}$. If $d \geqslant 1$ and d divides T, then $Tt(B, id)$ is (T/d)-automatic.*

Proof. Let $A = Tt(B, id)$. Denote by $h_0 < h_1 < h_2 < \cdots$ the strictly increasing sequence of the integers to which B assigns ω. Let $h_0 < h_1 < \cdots < h_{d-1}$ be the values of h_j which belong to $[0, T-1]$. It readily follows from the definition of A that A can be recursively defined by

$$\forall n \notin \{h_0, h_1, \cdots\} \quad A(n) = B(n)\,,$$
$$\forall n \geqslant 0\,, \quad A(h_n) = A(n)\,.$$

Moreover it is not hard to check that

$$\forall n \geqslant 0\,, \quad \forall j \in [0, d-1]\,, \quad h_{dn+j} = h_j + Tn\,.$$

Hence

$$\forall n \geqslant 0\,, \quad \forall j \in [0, d-1]\,, \quad \forall a \in [0, d-1] - \{h_0, h_1, \cdots, h_{d-1}\}\,,$$
$$A(h_j + Tn) = A(h_{dn+j}) = A(dn+j)\,,$$
$$A(a + Tn) = B(a + Tn) = B(a)\,.$$

Now let us define the set of sequences S with values in $\Gamma - \{\omega\}$ by: U is an element of S if and only if for every $j = 0, 1, \cdots, d-1$, the sequence $n \to U(dn+j)$ is either constant (hence the constant is an element of $\Gamma - \{\omega\}$), or of the form $n \to A(dn+k)$ for some $k = k(j) \in [0, d-1]$. Notice that the set S is finite as it contains at most $(\operatorname{Card}\Gamma - 1 + d)^d$ elements. Let $e = T/d$. To prove that the e-kernel of the sequence A is finite, it suffices to prove that the set S is stable under the maps $(X(n)) \to (X(en + r))$ for every r in $[0, e-1]$ (note that the sequences $n \to A(q^k n + r)$, $k \geqslant 0$, $0 \leqslant r \leqslant q^k - 1$ are obtained from the sequence $n \to A(n)$ by applying finitely many such maps, and that the sequence A itself belongs to S).

So let us take a sequence X in the set S, and an element r in $[0, e-1]$. Let $W(n) = X(en + r)$. To prove that W is in S, one computes $W(dn+j)$ for every j in $[0, d-1]$: $W(dn+j) = X(e(dn+j)+r) = X(d(en) + ej + r)$, (note that $ej + r \leqslant e(d-1) + e - 1 = T - 1$). Define a and b by $ej + r = ad + b$, with b in $[0, d-1]$, (hence a is in $[0, e-1]$). One has

$$W(dn+j) = X(den + ej + r) = X(d(en+a) + b)\,.$$

As X belongs to S, the sequence $(X(dn+b))$ is either constant or equal to the sequence $(A(dn+c))$ for a certain c in $[0, d-1]$. Hence $(X(d(en+a)+b))$ is either constant or equal to the sequence $(A(d(en+a)+c))) = (A(Tn + ad + c))$. But in turn the sequence $(A(Tn + ad + c))$ is either constant (if $ad + c$ is not one of the h_j's), or equal to the sequence $(A(Tn + h_u)) = (A(dn + u))$ (if $ad + c = h_u$, hence $u \leqslant d - 1$ because $ad + c \leqslant T - 1$).

Finally one has $(W(dn+j)) = (X(den+ej+r)) = (X(d(en+a)+b))$ is either constant or equal to the sequence $(A(dn+u))$, hence W belongs to S, which concludes the proof.

4) One can notice that the definition of $Tt(B, id)$ can be rewritten without supposing that B is periodic. One can then ask whether the Toeplitz transform of an automatic sequence is still automatic (see [1] for a particular case):

PROPOSITION. *Let* B *be a (non necessarily periodic) sequence on the alphabet* Γ *such that the set of* h *for which* $B(h) = \omega$ *is exactly the set* $\{qn + b;\ n \geqslant 0\}$, *where* q *and* b *are two natural numbers* ($q \geqslant 2$ *and* $0 < b < q$).

Then $Tt(B, id)$ *is* q*-automatic if and only if* B *is itself* q*-automatic.*

If the set $B^{-1}(\omega)$ *is not of the previous kind, the result need not hold.*

Proof. Let $A = Tt(B, id)$. Then, as usual,

$$\forall n \notin \{qk + b\ ;\ k \geqslant 0\} \qquad A(n) = B(n)\,,$$

$$\forall n \geqslant 0 \qquad A(qn+b) = A(n)\,.$$

From the first relation one has

$$\forall j \in [0, q-1] - \{b\}\,, \qquad A(qn+j) = B(qn+j)\,.$$

Moreover $B(qn+b) = \omega$.

If A is q-automatic, so are the sequences $(A(qn+j))$, hence so are all the sequences $(B(qn+j))$ for j in $[0, q-1] - \{b\}$; but the sequence $(B(qn+b))$ is constant, hence q-automatic. Finally all the sequences $(B(qn+j))$ are q-automatic for j in $[0, q-1]$, and this implies the q-automaticity of the sequence B itself.

If B is q-automatic, let K be its q-kernel (remember this is the set of subsequences $\{n \to B(q^k n + r);\ k \geqslant 0,\ 0 \leqslant r \leqslant q^k - 1\}$). K is finite. It is clear that the q-kernel of A is included in $K \cup \{A\}$, hence also finite, thus A is a q-automatic sequence.

Finally we give an example of a 2-automatic sequence which does not satisfy the condition on $B^{-1}(\omega)$, and for which $Tt(B, id)$ is not 2-automatic. Indeed define B by:

$$B(2^n) = \omega \quad \forall n \geqslant 1,$$

$$B(k) = 1 \quad \text{if } k \text{ is odd},$$

$$B(k) = 0 \quad \text{if } k \text{ is even and not in } \{2, 4, 8, 16, \cdots\}.$$

One easily computes

$$\begin{aligned}
B(2n) &= \omega && \text{if } n \in \{1, 2, 4, 8, \cdots\},\\
B(2n) &= 0 && \text{otherwise},\\
B(2n+1) &= 1 && \forall n \geqslant 0,\\
B(4n) &= B(2n) && \forall n \geqslant 0,\\
B(4n+2) &= \omega && \text{if } n = 0,\\
B(4n+2) &= 0 && \forall n \geqslant 1,\\
B(8n+2) &= B(4n+2) && \forall n \geqslant 0,\\
B(8n+6) &= 0 && \forall n \geqslant 0.
\end{aligned}$$

Hence the sequence B is 2-automatic (its kernel contains 5 elements). Note that $A = Tt(B, id)$ satisfies

$$\begin{aligned}
A(n) &= 1 && \text{if } n \text{ is odd},\\
A(2^n) &= A(n) && \forall n \geqslant 1,\\
A(n) &= 0 && \text{if } n \text{ is even and not in } \{2, 4, 8, \cdots\}.
\end{aligned}$$

If A were a 2-automatic sequence, it is well known that the sequence $A(2^n)$ would be ultimately periodic; hence from the second relation the sequence A itself would be ultimately periodic. Taking j large enough, and looking at $A(2^j+1), A(2^j+2), \cdots, A(2^{j+1}-1)$, one sees that A ends ultimately by $010101\cdots$ (or by $101010\cdots$, which is the same!). But for a huge odd number u one has $A(2^u) = A(u) = 1$, and $A(2^u+1) = 1$, which yields the desired contradiction.

Remark. A recent paper studies the ergodic properties of the generalized Rudin-Shapiro sequences (in the sense of [6]) using the Toeplitz device: A criterion for Toeplitz flows to be topologically isomorphic and applications, J. Kwiatkowski and Y. Lacroix, preprint, 1991.

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