5. Progression-free sequences and Toeplitz sequences

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If, instead of C one takes a "generalized" Morse sequence C', and if one defines

$$A'(n) = C'(n) + C'(n+1) \text{ modulo } 2 ,$$

then A' is also a Toeplitz sequence, as proved in [16].

4. Towers of Hanoi and Toeplitz sequences

The tower of Hanoi puzzle consists of three vertical pegs and of N circular disks of different diameters stacked in decreasing order on the first peg. At each step one may transfer the topmost disk from a peg to a different peg according to the rule: no disk is allowed to be on a smaller one. The game ends when all the disks are stacked on the second or third peg.

The sequence of moves for the classical (minimal) Hanoi tower algorithm can be generated in a very easy way as it is 2-automatic (see [4] and section 6), which essentially means that the k^{th} move can be predicted by a machine with bounded memory. More precisely number the pegs as I, II, III and define a (respectively b, c) to be the move which takes the topmost disk from peg I (respectively II, III) and puts it on peg II (respectively III, I). Let $\bar{a}, \bar{b}, \bar{c}$ be the respective opposite moves. Then the sequence of moves for N disks is the prefix of length $2^N - 1$ of an infinite sequence U which is 2-automatic. Moreover the following proposition is proved in [4]:

PROPOSITION. *The infinite sequence of moves U is equal to the Toeplitz transform of* $((a\bar{c}b\omega\, c\bar{b}a\omega\, b\bar{a}c\omega)^{\infty}, id)$.

Note that, keeping the notations of [4], the sequence U is indexed by $1, 2, \cdots$ and not by $0, 1, 2, \cdots$ as the sequences above.

5. Progression-free sequences and Toeplitz sequences

The question of finding a sequence of integers without arithmetic progressions of given length has been intensively studied (see its history in [14] and the included bibliography). In particular what is the "minimal" increasing sequence having this property?

One knows that, if k is a prime number, the minimal sequence of integers without any arithmetic progression of k terms is exactly the increasing sequence of the integers without the digit $k-1$ in their base k expansion (cited in [11], [26] and [12]). But nothing is known for the case where k is not prime (see [12]).

Let us define for every integer $k \geqslant 3$, $(U_k(n))$ as the increasing sequence of the integers without the digit $k-1$ in their base-k expansion. It is not difficult to obtain:

$$(*) \qquad \forall j \in [0, k-2]\,, \qquad U_k((k-1)n+j) = kU_k(n) + j\,.$$

If one considers the sequence of first differences of, say U_3, one obtains the sequence:

$$1 \quad 2 \quad 1 \quad 5 \quad 1 \quad 2 \quad 1 \quad 14 \quad 1 \quad 2 \quad 1 \quad 5 \quad 1 \quad 2 \quad 1 \quad \cdots$$

This sequence resembles somewhat the paperfolding sequence, (except that it takes infinitely many values), which gives the idea of the following easy proposition:

PROPOSITION. *Let k be an integer greater than or equal to 3, define the sequence $(U_k(n))$ by $(*)$. Let $D_k(n) = U_k(n+1) - U_k(n)$. Finally let g_k be defined on $\mathbf{N} \cup \{\omega\}$ by $g_k(x) = kx - k + 2$ if x is in $\mathbf{N}$ and $g_k(\omega) = \omega$.*

Then

$$D_k = Tt((1^{k-2}\omega)^\infty, g_k)\,,$$

(see notations in paragraph 1).

Proof. From the definition of U_k, one has

$$D_k((k-1)n+j) = 1 \quad \text{for every } j \text{ in } [0, k-3] \text{ and every integer } n\,,$$

$$D_k((k-1)n+k-2) = kD_k(n) - (k-2) = g_k(D_k(n)) \quad \text{for every integer } n\,.$$

Remark. For a very curious occurrence of the sequence U_k see [19].

6. MISCELLANEOUS QUESTIONS

In this paragraph we first give some other examples of naturally occurring Toeplitz sequences. Second we shall study the connections with automatic sequences.

1) Among other examples of Toeplitz transforms let us give three natural sequences:

— Let p be a prime number, and $v_p(n)$ be the highest power of p dividing n. Let $U(n) = v_p(n+1)$, and let f be the function defined